28) Ukažte, že funkce $S(q,t,Q) = m\omega \dfrac{(q^2+Q^2)\cos(\omega t) - 2qQ}{2\sin(\omega t)}$ je při $0<t<\pi$ úplným integrálem (hlavní funkcí Hamiltonovou) HJR pro LHO $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$ a najděte fázové trajektorie.

$\underline{H + \frac{\partial S}{\partial t} = 0}$ $\quad \underline{p = \frac{\partial S}{\partial q}}$ $(\Leftarrow F_1)$

HJR $\quad \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0 \qquad \frac{1}{2m}\left(m\omega \frac{2q\cos(\omega t) - 2Q}{2\sin(\omega t)}\right)^2 + \frac{1}{2}m\omega^2 q^2 + m\omega\left[\frac{(q^2+Q^2)(-\sin\omega t)\omega \cdot 2\sin(\omega t) - 2\omega\cos\omega t\,((q^2+Q^2)\cos\omega t - 2qQ)}{(2\sin(\omega t))^2}\right]$

$$= \frac{\omega^2 m}{2}\frac{(q\cos\omega t - Q)^2}{\sin^2\omega t} + \frac{m\omega^2}{2}q^2 + m\omega \frac{-(q^2+Q^2)2\omega \sin^2\omega t - 2\omega((q^2+Q^2)\cos^2\omega t - 2qQ\cos\omega t))}{4\sin^2\omega t} =$$

$$= \frac{\omega^2 m}{2\sin^2\omega t}\left(q^2\cancel{\cos^2\omega t} - \cancel{2qQ\cos\omega t} + \cancel{Q^2} + q^2\underline{\sin^2\omega t} - (q^2 + \cancel{Q^2})\cdot 1 + \cancel{2qQ\cos\omega t}\right) = 0 \checkmark$$

$$P = -\frac{\partial S}{\partial Q}(\underline{q},t,Q) = m\omega \frac{2Q\cos\omega t - 2q}{2\sin\omega t} \rightarrow \underline{q(t,Q,P) = -\frac{P}{m\omega}\sin(\omega t) + Q\cos\omega t}$$

Fázové Trajektorie

$$\underline{p} = \frac{\partial S}{\partial q} = m\omega \frac{2q\cos\omega t - 2Q}{2\sin\omega t} = \frac{m\omega}{\sin\omega t}\left(-\frac{P}{m\omega}\sin\omega t\cos\omega t + \underbrace{Q\cos^2\omega t - Q}_{-\sin^2 t}\right) = \underline{-P\cos\omega t - m\omega Q\sin\omega t}$$

Teorém Noetherové pro $F = F(\vec{q},\vec{p})$ $\quad \boxed{F \text{ je I.P} \Leftrightarrow \{F,H\} = 0}$

29) Ukažte, že veličina $L_3 = q_1p_2 - q_2p_1$ je generátorem rotace.

$$dL_3 = q_1 dp_2 + \underset{a}{p_2 dq_1} - q_2 dp_1 - \underset{T_0^*\Gamma}{p_1 dq_2}$$

$\omega = dp_i \wedge dq_i$

$dp_i \leftrightarrow \frac{\partial}{\partial q_i}$

$dq_i \leftrightarrow -\frac{\partial}{\partial p_i}$

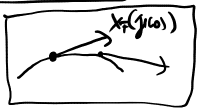
$$X_{L_3} = q_1\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial q_1} - p_2\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial p_2}$$

Báze $T_0\Gamma$

$$\boxed{F \rightarrow \vec{X}_F = \begin{pmatrix} \frac{\partial F}{\partial p_i} \\ -\frac{\partial F}{\partial q_i}\end{pmatrix}} = \begin{pmatrix} -q_2 \\ q_1 \\ -p_2 \\ p_1 \end{pmatrix}$$

Hamiltonovské Vek. pole

Integrální křivky X_F $\quad \boxed{\frac{d\gamma(\varepsilon)}{d\varepsilon} = X_F(\gamma(\varepsilon))}$



$$q_1' = \frac{dq_1}{d\varepsilon} = \frac{\partial L_3}{\partial p_1} = -q_2 \qquad \frac{dp_1}{d\varepsilon} = -\frac{\partial L_3}{\partial q_1} = -p_2$$

$$q_2' = \frac{dq_2}{d\varepsilon} = \frac{\partial L_3}{\partial p_2} = q_1 \qquad \frac{dp_2}{d\varepsilon} = -\frac{\partial L_3}{\partial q_2} = p_1$$

$q_1' = -q_2 \quad / \frac{d}{d\varepsilon} \quad q_1'' = -q_2' = -q_1 \qquad \underline{q_1'' + q_1 = 0}$

$q_2' = q_1$

$q_2 = -q_1'$

$\hookrightarrow q_1(\varepsilon) = C\sin(\varepsilon + D) = \underline{A}\cos\varepsilon + \underline{B}\sin\varepsilon$

$q_1(0) = A$

$q_2(\varepsilon) = -q_1'(\varepsilon) = +A\sin\varepsilon - B\cos\varepsilon$

$q_2(0) = -B$

$$\boxed{\begin{aligned} q_1(\varepsilon) &= q_1(0)\cos\varepsilon - q_2(0)\sin\varepsilon \\ q_2(\varepsilon) &= q_1(0)\sin\varepsilon + q_2(0)\cos\varepsilon \end{aligned}}$$

$Q_1 \quad q_1$

$Q_2 \quad q_2$

Transformace pro $s=2$

$$Q_1^\varepsilon = q_1\cos\varepsilon - q_2\sin\varepsilon$$
$$Q_2^\varepsilon = q_1\sin\varepsilon + q_2\cos\varepsilon$$

pro $s=3$ $\quad Q_3^\varepsilon = q_3$

Rotace//

obdobně $\quad P_1^\varepsilon = p_1\cos\varepsilon - p_2\sin\varepsilon$

$P_2^\varepsilon = p_1\sin\varepsilon + p_2\cos\varepsilon$

$P_3^\varepsilon = p_3$

Pozn $\quad p_i \rightarrow X_{p_i} = \frac{\partial}{\partial q_i}$ je generátor translace v q_i $\quad Q_i = q_i + \varepsilon \quad Q_j = q_j \ \forall j \neq i$

Taylor do 1. řádu v ε

Infinitesimální Tr

$$\frac{dq_1}{d\varepsilon} = \underline{-q_2} \rightarrow \underline{q_1(\varepsilon) = q_1(0) + \varepsilon \left.\frac{dq_1}{d\varepsilon}\right|_{\varepsilon=0}} = q_1(0) + \varepsilon(-q_2(0)) = \underline{q_1(0) - \varepsilon q_2(0)}$$

$$\frac{dq_2}{d\varepsilon} = \underline{q_1} \rightarrow \underline{q_2(\varepsilon) = q_2(0) + \varepsilon \left.\frac{dq_2}{d\varepsilon}\right|_{\varepsilon=0}} = \underline{q_2(0) + \varepsilon q_1(0)}$$

30) Ukažte, že funkce $G(q_{\alpha i}, p_{\alpha i}, t) = \sum_{\alpha=1}^{N} m_\alpha q_{\alpha 1} - t\sum_{\alpha=1}^{N} p_{\alpha 1}$ je generátorem speciální Galileiho transformace na fázovém prostoru dimenze 6N pro soustavu N částic.

$$G \xrightarrow{d} dG = \sum_\alpha{}' \Big(m_\alpha \, dq_{\alpha 1} - \underbrace{t}_{\text{parametr } (dt=0)} dp_{\alpha 1} \Big) \xrightarrow[\substack{dq \leftrightarrow -\frac{\partial}{\partial p} \\ dp \leftrightarrow \frac{\partial}{\partial q}}]{\omega} X_G = \sum_\alpha{}' \Big(m_\alpha \big(-\frac{\partial}{\partial p_{\alpha 1}}\big) - t \frac{\partial}{\partial q_{\alpha 1}} \Big) \quad \vec{X}_G = \begin{pmatrix} -t \\ 0 \\ 0 \\ -t \\ 0 \\ 0 \\ \vdots \\ -m_1 \\ 0 \\ 0 \\ -m_2 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{array}{l} \}\alpha=1 \\ \\ \\ \}\alpha=2 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

GN

$$\frac{dq_{\alpha 1}}{d\varepsilon} = \frac{\partial G}{\partial p_{\alpha 1}} = -t \quad \Rightarrow \quad q_{\alpha 1}(\varepsilon) = -t\varepsilon + q_{\alpha 1}(0) \quad \rightarrow \quad q_{\alpha 1} = Q_{\alpha 1} + t \cdot V$$

$$\frac{dq_{\alpha j}}{d\varepsilon} = \frac{\partial G}{\partial p_{\alpha j}} = 0 \quad j = 2,3 \qquad q_{\alpha j}(\varepsilon) = q_{\alpha j}(0) \qquad q_{\alpha j} = Q_{\alpha j} \qquad \forall \alpha \in \hat{N}$$

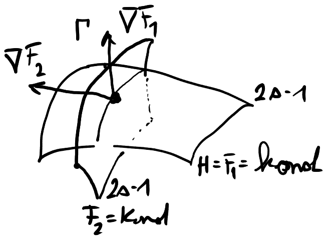
$\forall \alpha \in \hat{N}$ $\qquad \varepsilon = V$ rychlost soustavy

$$\frac{dp_{\alpha 1}}{d\varepsilon} = -\frac{\partial G}{\partial q_{\alpha 1}} = -m_\alpha \quad \Rightarrow \quad p_{\alpha 1}(\varepsilon) = -m_\alpha \varepsilon + p_{\alpha 1}(0) \quad \rightarrow \quad p_{\alpha 1} = P_{\alpha 1} + m_\alpha V$$

$$p_{\alpha j}(\varepsilon) = p_{\alpha j}(0) \qquad p_{\alpha j} = P_{\alpha j}$$

$$\frac{dp_{\alpha j}}{d\varepsilon} = -\frac{\partial G}{\partial q_{\alpha j}} = 0 \quad j = 2,3$$

31) Kolik navzájem nezávislých integrálů pohybu tvaru $F_i(\vec{q},\vec{p})$ (tj. nezávisejících na čase) může mít soustava o s stupních volnosti popsaná Hamiltonovou funkcí $H(\vec{q},\vec{p})$?



$2s-2$, $2s-3$, 1

1) $\frac{\partial H}{\partial t} = 0 \Rightarrow H$ je I.P. $\quad H = F_1 = \boxed{\text{konst.} = H(\vec{q},\vec{p})}$ je nadplocha v Γ

2) $F_2(\vec{q},\vec{p}) = \text{konst} = \alpha_2 \rightarrow$ definuje nadplochu

3) $F_3(\vec{q},\vec{p}) = \alpha_3 = \text{konst.}$

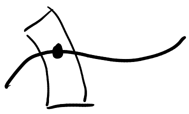
⋮

2s-1) $F_{2s-1}(\vec{q},\vec{p}) = \text{konst}$ $\quad \rightarrow$ průnik $2s-1$ "nezávislých" nadploch je křivka (1-dim. varieta)

⇓

Tvar fázové trajektorie

$\cancel{F_{2s} = \alpha_{2s}}$



$F_{2s}(\vec{q},\vec{p},t) = \alpha = \text{konst} \rightarrow$ časová parametrizace trajektorie

I.P. jsou nezávislé pokud $(\nabla F_1, \dots, \nabla F_k)$ jsou LN soubor na Γ

Pozn. Pokud známe $2s$ nezávislých I.P. $\Rightarrow$ pomocí algebraických úprav najdeme fázovou trajektorii

32) Definice integrabilní soustavy, Liouvilleova věta o integrabilních soustavách.

Soustava o s-stupních volnosti s Hamiltoniánem $H=H(\vec{q},\vec{p})$ se nazývá integrabilní, když má s nezávislých globálních integrálů pohybu $P_j(\vec{q},\vec{p})$ $j\in\hat{s}$ v involuci tj.

1) $\{P_j, H\} = 0 \quad \forall j \in \hat{s}$

2) $\{P_i, P_j\} = 0 \quad \forall i,j \in \hat{s}$

3) $(\nabla P_1, \dots, \nabla P_s)$ jsou LN na Γ /(bez singulárních bodů)

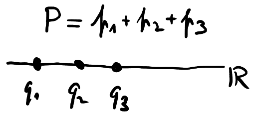
Liouvilleova věta: Pohybové rce. integrabilní soustavy jsou řešitelné kvadraturami.

33) Todova molekula. Ukažte, že lineární tříatomová molekula s Hamiltoniánem $H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+e^{q_1-q_2}+e^{q_2-q_3}+e^{q_3-q_1}$ je integrabilní soustava. Návod: zkoumejte první integrály H,P,K.

$\frac{\partial H}{\partial t}=0 \Rightarrow H$ je I.P.

$P = p_1+p_2+p_3$

$$K = -\frac{1}{9}\underbrace{(p_1+p_2-2p_3)}_{A}\underbrace{(p_2+p_3-2p_1)}_{B}\underbrace{(p_3+p_1-2p_2)}_{C} + Ae^{q_1-q_2} + Be^{q_2-q_3} + Ce^{q_3-q_1}$$



$$\{H,P\} = \frac{\partial H}{\partial q_i}\underbrace{\frac{\partial P}{\partial p_i}}_{=1} - \frac{\partial H}{\partial p_i}\underbrace{\frac{\partial P}{\partial q_i}}_{=0} = \sum_{i=1}^{3}\frac{\partial H}{\partial q_i} = (\cancel{e^{q_1-q_2}} - \cancel{e^{q_3-q_1}})\cdot 1 + (-\cancel{e^{q_1-q_2}} + e^{q_2-q_3})\cdot 1 + (-e^{q_2-q_3} + \cancel{e^{q_3-q_1}})\cdot 1 = 0$$

$$\{K,P\} = \frac{\partial K}{\partial q_i}\frac{\partial P}{\partial p_i} - \frac{\partial K}{\partial p_i}\frac{\partial P}{\partial q_i} = \sum_{i=1}^{3}\frac{\partial K}{\partial q_i} = A\cancel{e^{q_1-q_2}} - \cancel{Ce^{q_3-q_1}} - \cancel{Ae^{q_1-q_2}} + Be^{q_2-q_3} - Be^{q_2-q_3} + \cancel{Ce^{q_3-q_1}} = 0$$

$$\{K,H\} = \frac{\partial K}{\partial q_i}\underbrace{\frac{\partial H}{\partial p_i}}_{=p_i} - \frac{\partial K}{\partial p_i}\frac{\partial H}{\partial q_i} = (Ae^{q_1-q_2} - Ce^{q_3-q_1})p_1 + (-Ae^{q_1-q_2} + Be^{q_2-q_3})p_2 + (-Be^{q_2-q_3} + Ce^{q_3-q_1})p_3 - \left[\frac{-1}{9}(BC-2AC+AB) + e^{q_1-q_2}\right.$$
$$\left. - 2e^{q_2-q_3} + e^{q_3-q_1}\right](e^{q_1-q_2} - e^{q_3-q_1}) - \left[\frac{-1}{9}(BC+AC-2AB) + e^{q_1-q_1} + e^{q_2-q_3} - 2e^{q_3-q_1}\right](-e^{q_1-q_2} + e^{q_2-q_3})$$
$$- \left[\frac{-1}{9}(-2BC+AC+AB) - 2e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3-q_1}\right](-e^{q_2-q_3} + e^{q_3-q_1}) =$$

$$= e^{q_1-q_2}\Big(\cancel{Ap_1} - Ap_2 + \frac{1}{9}\overbrace{\cancel{(-3AC+3AB)}}^{3(p_2-p_1)A} - 3e^{q_2-q_3} + 3e^{q_3-q_1}\Big) + e^{q_2-q_3}\Big(Bp_2 - \cancel{Bp_3} + \frac{1}{9}\underbrace{\cancel{(-3AB+3BC)}}_{3B(3p_3-3p_2)} - 3e^{q_3-q_1} + 3e^{q_1-q_2}\Big)$$
$$+ e^{q_3-q_1}\Big(\cancel{Cp_3} - \cancel{Cp_1} + \frac{1}{9}\underbrace{(-3BC+3CA)}_{3C(\underbrace{A-B}_{\cancel{3p_1-3p_3}})} - 3e^{q_1-q_2} + 3e^{q_2-q_3}\Big) = 0$$

$$(\nabla P, \nabla H, \nabla K) = \begin{pmatrix} 0 & e^{q_1-q_2}-e^{q_3-q_1} & \\ 0 & \cdot & \\ 0 & \cdot & \\ 1 & p_1 & \\ 1 & p_2 & \\ 1 & p_3 & \end{pmatrix}$$

"jsou" LN

$p_1=p_2=p_3$ $\boxed{q_1=q_2=q_3}$ ×

$\Rightarrow$ je integrabilní soustava!