43. Ukažte, že Lorentzova síla je zobecněná potenciální síla. $U = q(\varphi(\vec x,t) - \vec v\cdot\vec A(\vec x,t)) = U(\vec x,\vec v,t)$, $\vec v = \dot{\vec x}$

$$F_i = -\frac{\partial U}{\partial x_i} + \frac{\hat d}{dt}\left(\frac{\partial U}{\partial \dot x_i}\right) \qquad (\nabla_v)_i = \frac{\partial}{\partial v_i} \equiv \frac{\partial}{\partial \dot x_i} \qquad \vec E = -\nabla\varphi - \frac{\partial \vec A}{\partial t} \qquad \vec B = \nabla\times\vec A$$

$$\Updownarrow$$

$$\vec F = -\nabla U + \frac{d}{dt}(\nabla_v U) = q\Big[-\nabla\varphi + \underbrace{\nabla(\vec v\cdot\vec A)}_{2.\,\text{cv.}} + \frac{d}{dt}\Big(\nabla_v\varphi - \underbrace{\nabla_v(\vec v\cdot\vec A)}\Big)\Big] = q\Big[-\nabla\varphi + (\vec v\nabla)\vec A + \cancel{(\vec A\nabla)\vec v}$$

$$+ \vec v\times(\nabla\times\vec A) + \vec A\times\cancel{(\nabla\times\vec v)} + \frac{d}{dt}\Big(\cancel{\nabla_v\varphi} - \big[\cancel{(\vec v\nabla_v)\vec A} + (\vec A\nabla_v)\vec v + \cancel{\vec v\times(\nabla_v\times\vec A)} + \vec A\times\cancel{(\nabla_v\times\vec v)}\big]\Big)\Big] = * \qquad A_j\frac{\partial}{\partial x_j}v_i = 0$$

$$\varepsilon_{ijk}\frac{\partial}{\partial x_j}v_k = 0 \qquad \frac{\partial}{\partial v_i}\varphi = 0 \qquad v_j\frac{\partial A_i}{\partial v_j} = 0 \qquad \nabla_v\times\vec A = 0 \qquad A_j\frac{\partial v_i}{\partial v_j} = A_j\delta_{ij} = A_i \qquad \varepsilon_{ijk}\frac{\partial v_k}{\partial v_j} = \varepsilon_{ijk}\delta_{kj} = 0$$

antisym. sym.

$$* = q\Big[-\nabla\varphi + (\vec v\nabla)\vec A + \vec v\times(\nabla\times\vec A) - \frac{d}{dt}(\vec A)\Big] = q\Big[-\nabla\varphi + \cancel{(\vec v\nabla)\vec A} + \vec v\times(\nabla\times\vec A) - \cancel{(\vec v\nabla)\vec A} - \frac{\partial\vec A}{\partial t}\Big] =$$

$$\left(\frac{d\vec A}{dt}\right)_i = \frac{dA_i}{dt} = \frac{\partial A_i}{\partial x_j}\underbrace{\dot x_j}_{= v_j} + \frac{\partial A_i}{\partial t} = v_j\frac{\partial A_i}{\partial x_j} + \frac{\partial A_i}{\partial t} = \left((\vec v\nabla)\vec A + \frac{\partial\vec A}{\partial t}\right)_i$$

$$\vec F = q\Big[\underbrace{-\nabla\varphi - \frac{\partial\vec A}{\partial t}}_{\vec E} + \vec v\times\underbrace{(\nabla\times\vec A)}_{\vec B}\Big] = \underline{\underline{q\big[\vec E + \vec v\times\vec B\big]}}$$

44. Jak se liší Lagrangeovy funkce pro nabitý hm. bod v elektromagnetickém poli pro různé kalibrace? Ukažte, že tento rozdíl nemá vliv na Lagrangeovy rovnice.

$$\vec A' = \vec A + \text{grad}\,\Lambda \qquad \Lambda = \Lambda(\vec x,t)$$
$$\varphi' = \varphi - \frac{\partial\Lambda}{\partial t} \qquad \text{třídy } C^{(2)}$$

$$L = T - U = \frac12 m\vec v^2 - q(\varphi - \vec v\cdot\vec A)$$

$$L' = \frac12 m\vec v^2 - q(\varphi' - \vec v\cdot\vec A') = \frac12 m\vec v^2 - q\Big(\varphi - \underbrace{\frac{\partial\Lambda}{\partial t}} - \vec v\cdot\vec A - \underbrace{\vec v\cdot\nabla\Lambda}\Big) = \underbrace{\frac12 m\vec v^2 - q(\varphi - \vec v\cdot\vec A)}_{L} + q\underbrace{\left(\frac{\partial\Lambda}{\partial t} + v_j\frac{\partial\Lambda}{\partial x_j}\right)}_{\frac{d\Lambda}{dt}}$$

$$L' = L + \frac{d(q\Lambda)}{dt} \qquad \boxed{f = q\Lambda(\vec x,t)}$$

LR1D. $\forall i = 1,2,3$

$$\boxed{0 = Q_j^{(nep)}} = \frac{d}{dt}\left(\frac{\partial L'}{\partial\dot x_i}\right) - \frac{\partial L'}{\partial x_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot x_i} + \underbrace{\frac{\partial}{\partial\dot x_i}\frac{df}{dt}}\right) - \frac{\partial L}{\partial x_i} - \underbrace{\frac{\partial}{\partial x_i}\frac{df}{dt}} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot x_i}\right) - \frac{\partial L}{\partial x_i} + \frac{d}{dt}\left(\frac{\partial f}{\partial x_i}\right) - \left(\frac{\partial^2 f}{\partial x_i\partial x_j}\dot x_j + \frac{\partial^2 f}{\partial x_i\partial t}\right) = (*)$$

třída $C^{(2)}$

$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial x_j}\dot x_j} + \frac{\partial f}{\partial t} \qquad \frac{\partial}{\partial\dot x_i}\left(\frac{df}{dt}\right) = \frac{\partial f}{\partial x_j}\frac{\partial\dot x_j}{\partial\dot x_i} + 0 = \frac{\partial f}{\partial x_j}\delta_{ij} = \frac{\partial f}{\partial x_i}$$

$$(*) = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot x_i}\right) - \frac{\partial L}{\partial x_i} + \cancel{\frac{\partial^2 f}{\partial x_j\partial x_i}\dot x_j} + \cancel{\frac{\partial^2 f}{\partial t\partial x_i}} - \cancel{\frac{\partial^2 f}{\partial x_j\partial x_i}\dot x_j} - \cancel{\frac{\partial^2 f}{\partial t\partial x_i}} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot x_i}\right) - \frac{\partial L}{\partial x_i}$$

Lagrangeův formalizmus pro N částic v $\mathbb{R}^3$

1) Volba kartézských souřadnic v inerciální soustavě

2) Lagrangeova funkce v kartézských souřadnicích $L = T - U = \overbrace{\frac12\sum_{i=1}^{3N} m_i\dot x_i^2}^{T} - U(\vec x,\dot{\vec x},t)$ $\qquad T = \frac12\sum_{\alpha=1}^{N}\sum_{j=1}^{3} m_\alpha\dot x_{\alpha j}^2$

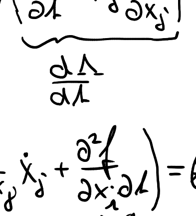
3) Vazby - nezávislé holonomní vazby $f_k(\vec x,t) = 0 \quad k\in\hat r$

def. konfigurační prostor $M(t) = \{\vec x\in\mathbb{R}^{3N} \mid f_k(\vec x,t) = 0\ \forall k\in\hat r\}$

$\dim M = \boxed{3N - r = s}$ počet stupňů volnosti

obecné rce. konf. pr.

nezávislost vazeb: $(\nabla f_1,\dots,\nabla f_r)$ tvoří LN soubor $\forall t\ \forall\vec x\in M$



$f(x) = x \le 0$

4) obecné souřadnice q_j $j\in\hat s$ volíme tak aby splnily vazby $\hat f_k(\vec q,t) = f_k(\vec X(\vec q,t),t) \equiv 0\ \forall t\ \forall\vec q$

$x_i = \hat X_i(\vec q,t)$ ← parametrické rce. konf. pr.

$$\dot x_i = \dot{\hat X}_i = \frac{\hat d\hat X_i}{dt} = \frac{\partial\hat X_i}{\partial q_j}\dot q_j + \frac{\partial\hat X_i}{\partial t}$$

5) Lagrangeova funkce v obecných souřadnicích $\hat L = \hat L(\vec q,\dot{\vec q},t) = L(\vec X(\vec q,t),\dot{\vec X}(\vec q,\dot{\vec q},t),t)$

6) Lagr. rc. s obecných

$$\forall i \in \hat{s} \quad \boxed{\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_i}\right) - \frac{\partial \hat{L}}{\partial q_i} = Q_i^{(nep)} = 0}$$

většinou

$$Q_i^{(nep)} = F_j^{(nep)} \frac{\partial \hat{x}_j}{\partial q_i}$$ obecné nepotenciální síly

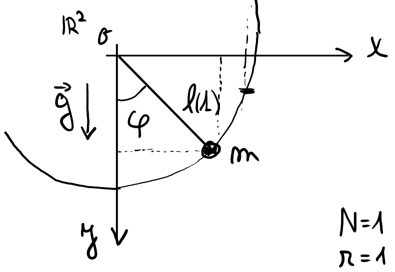
Obecná síla

$$Q_j = \frac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$$

---

7) obecná hybnost $p_j = \frac{\partial \hat{L}}{\partial \dot{q}_j}$ obecná energie $E = \left(\sum_{j=1}^{\hat{s}} \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) - L$ $\quad [\dot{p}_i = Q_i]$

46. Odvoďte pohybovou rovnici pro matematické kyvadlo s délkou závěsu $l(t) = l_0(1+kt)$ $l_0, k > 0$ konst



1) osy ✓

2) $L(\vec{x},\dot{\vec{x}},t) = T - U = \frac{1}{2}m(\overbrace{\dot{x}^2+\dot{y}^2}^{\vec{v}^2}) + mgy$ $\quad U = -mgy$ $\quad \vec{F}_g = -\nabla U$

3) Vazby $f(x,y,t) = x^2+y^2-l^2(t) = 0$ ← holonomní, reonomní (t), udržující (=), ideální

Konf. pr. kružnice se středem O a poloměrem $l(t)$ $\quad N=1$, $r=1$

$\boxed{s = 1\cdot 2 - 1 = 1}$

4) obecné souřadnice φ

$x = l(t)\sin\varphi \qquad \dot{x} = \dot{l}(t)\sin\varphi + l(t)\dot{\varphi}\cos\varphi$

$y = l(t)\cos\varphi \qquad \dot{y} = \dot{l}(t)\cos\varphi - l(t)\dot{\varphi}\sin\varphi$

5) $L = \frac{1}{2}m\left(\dot{l}^2(t) + l^2(t)\dot{\varphi}^2\right) + mgl(t)\cos\varphi$

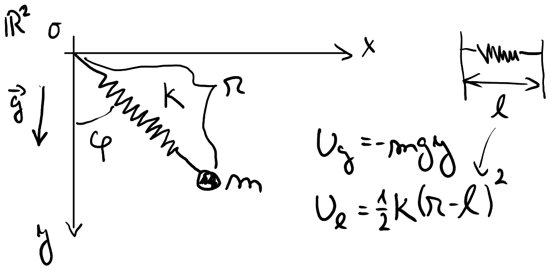
6) Lagr. rc $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$

$$\frac{d}{dt}\left(\frac{1}{2}m\, l^2\, 2\dot{\varphi}\right) - (-mgl\sin\varphi) = \ddot{\varphi}\, l^2 m + \underbrace{2l\dot{l}m\dot{\varphi}} + mgl\sin\varphi = 0$$

$$\ddot{\varphi} + \frac{2\dot{l}}{l}\dot{\varphi} + \frac{g}{l}\sin\varphi = 0 \qquad \dot{l} = l_0 k$$

$$\boxed{\ddot{\varphi} + \frac{2l_0k}{l_0(1+kt)}\dot{\varphi} + \frac{g}{l_0(1+kt)}\sin\varphi = 0}$$

42. Odvoďte pohybové rovnice pro matematické kyvadlo s pružným závěsem tuhosti k a nezatíženou délkou l.



$U_g = -mgy$

$U_l = \frac{1}{2}k(r-l)^2$

① ✓

② $L = T - U = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + mgy - \frac{1}{2}k\left(\sqrt{x^2+y^2} - l\right)^2$

③ Vazby $\emptyset$ $r=0$ $\boxed{s = 1\cdot 2 - 0 = 2}$ $\quad$ Konf. pr. celé $\mathbb{R}^2$

④ obecné souřadnice $\boxed{r, \varphi}$ $\quad$ (může brát x, y)

$x = r\sin\varphi \qquad \dot{x} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$

$y = r\cos\varphi \qquad \dot{y} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$

⑤ $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + mgr\cos\varphi - \frac{1}{2}k(r-l)^2$

⑥ Lagr. rc $(s=2)$

(1) φ: $0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(mr^2\dot{\varphi}\right) - (-mgr\sin\varphi) = mr^2\ddot{\varphi} + 2mr\dot{r}\dot{\varphi} + mgr\sin\varphi$

(2) r: $0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{dt}(m\dot{r}) - \left(mr\dot{\varphi}^2 + mg\cos\varphi - k(r-l)\right) = m\ddot{r} - mr\dot{\varphi}^2 - mg\cos\varphi + k(r-l)$

Zkoumejte limitu pro $\frac{k}{m} \to +\infty$ $\left(\Leftrightarrow \frac{m}{k} \to 0\right)$ $\quad$ (2) $0 = \frac{m}{k}\left(\ddot{r} - \cancel{r\dot{\varphi}^2} - g\cos\varphi\right) + r - l \to$ Vazby $\boxed{r - l = 0}$

$\frac{m}{k} \to 0$

$r = l$

(1) $r = l$ $\quad \dot{r} = 0$ $\qquad ml^2\ddot{\varphi} + mgl\sin\varphi = 0$

$$\boxed{\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0}$$ Mat. kyvadlo.