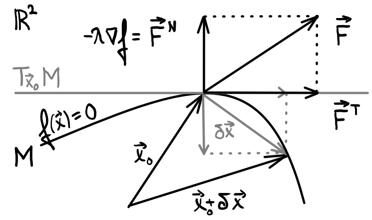
Posunutí ve shodě s vazbou $f(\vec{x})=0$ $\quad \underbrace{f(\vec{x}_0+\delta\vec{x})}_{=0} = \underbrace{f(\vec{x}_0)}_{=0} + \underbrace{\frac{\partial f(\vec{x}_0)}{\partial \vec{x}}}_{\nabla f(\vec{x}_0)} \cdot \delta\vec{x} + \underbrace{\mathcal{O}(|\delta\vec{x}|^2)}_{\doteq 0}$

malé posunutí (infinitezimální)



Virtuální posunutí – myšlené okamžité infinitezimánlní posunutí ve shodě s vazbou

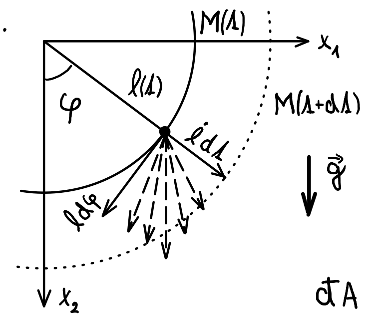
Jsou–li všechny vtištěné síly konzervativní $\vec{F}(\vec{x}) = -\nabla U(\vec{x})$ pak $0 = \delta A = (-\nabla U + \lambda_k \nabla f_k)\cdot \delta\vec{x} = -\delta(U - \underbrace{\lambda_k f_k}) = -\delta\tilde{U}$
a úloha se redukuje na hledání vázaného extrému funkce $U(\vec{x})$ vzhledem k varietě M "potenciální energie vazebných sil"
t. j. $\delta\tilde{U}(\vec{x}_0) = 0 \quad f_k(\vec{x}_0) = 0 \ \forall k \in \hat{n} \qquad \delta^2\tilde{U}(\vec{x}_0)\Big/_{T_{\vec{x}_0}M}$ >0 stabilní, <0 labilní, $=0$ indiferentní

<u>Infinitezimální posunutí</u> holonomní soustavy s rheonomními vazbami $\quad f_k(\vec{x},t) = 0 \ \forall k \in \hat{n} \qquad \vec{x} = \hat{\vec{x}}(\vec{q},t)$

- možné (reálné) $\quad 0 = df_k = \frac{\partial f_k}{\partial x_i} dx_i + \frac{\partial f_k}{\partial t} dt \qquad dx_i = \frac{\partial \hat{x}_i}{\partial q_j} dq_j + \frac{\partial \hat{x}_i}{\partial t} dt \qquad f(\vec{x}+d\vec{x}, t+dt) = f(\vec{x},t) + \frac{\partial f}{\partial \vec{x}}\cdot d\vec{x} + \frac{\partial f}{\partial t} dt + \dots$
- virtuální $\quad 0 = \delta f_k = \frac{\partial f_k}{\partial x_i}\delta x_i \qquad \delta x_i = \frac{\partial \hat{x}_i}{\partial q_j}\delta q_j \qquad f(\vec{x}+\delta\vec{x}, t) = f(\vec{x},t) + \frac{\partial f}{\partial \vec{x}}\cdot\delta\vec{x} + \dots$
- skutečné $\quad 0 = d\tilde{f}_k = \left(\frac{\partial f_k}{\partial x_i}\dot{x}_i + \frac{\partial f_k}{\partial t}\right)dt \qquad d\tilde{x}_i = \left(\frac{\partial \hat{x}_i}{\partial q_j}\dot{q}_j + \frac{\partial \hat{x}_i}{\partial t}\right)dt$

lze získat z možného posunutí dosazením trajektorie $x_i = \tilde{x}_i(t),\ q_j = \tilde{q}_j(t)$

Př.



$f(x_1,x_2,t) = x_1^2 + x_2^2 - l^2(t) = 0$

$x_1 = l(t)\sin\varphi \qquad \vec{F} = \begin{pmatrix} 0 \\ mg \end{pmatrix}$
$x_2 = l(t)\cos\varphi$

$\downarrow \vec{g}$

Virtuální posunutí a vitruální práce

$0 = \delta f = 2x_1\delta x_1 + 2x_2\delta x_2 \qquad \delta\vec{x} = \begin{pmatrix}\cos\varphi \\ -\sin\varphi\end{pmatrix} l\,\delta\varphi$

$\delta A = (\vec{F} + \lambda\nabla f)\cdot\delta\vec{x} = \vec{F}\cdot\delta\vec{x} + \underbrace{\lambda\nabla f\cdot\delta\vec{x}}_{=0} = 0\cdot\delta x_1 + (mg)\delta x_2$

virtuální práce vazebných sil

Možné posunutí a práce

$0 = df = 2x_1 dx_1 + 2x_2 dx_2 - 2l\dot{l}dt \qquad d\vec{x} = \begin{pmatrix}\cos\varphi \\ -\sin\varphi\end{pmatrix} l\,d\varphi + \begin{pmatrix}\sin\varphi \\ \cos\varphi\end{pmatrix}\dot{l}\,dt$

skutečná práce

$dA = (\vec{F}+\lambda\nabla f)\cdot d\vec{x} = \vec{F}\cdot d\vec{x} + \lambda\nabla f\cdot d\vec{x} = 0\cdot dx_1 + mg\,dx_2 + 2\lambda x_1 dx_1 + 2\lambda x_2 dx_2 = mg\,dx_2 + \overbrace{\lambda 2l\dot{l}dt}$

výkon vazebné síly $= -\lambda\frac{\partial f}{\partial t}$

Př. pro $l = konst.$ $0 = \delta A = mg\delta x_2 = -mg\underbrace{\sin\varphi}_{=0}\delta\varphi \Rightarrow \varphi = 0,\pi$ je rovnovážná poloha

2, d'Alembertův princip – dynamická rovnováha soustavy částic podrobených vazbám $f_k(\vec{x},t) = 0 \ \forall k \in \hat{n}$

(LR1D) $\quad \mathbb{M}\ddot{\vec{x}} = \vec{F}(\vec{x},\dot{\vec{x}},t) + \vec{R}(\vec{x},\dot{\vec{x}},t) = \vec{F}(\vec{x},\dot{\vec{x}},t) + \sum_{k=1}^{n}\lambda_k\nabla f_k$

$\mathbb{M} = diag(m_1,\dots,m_{3N}) \qquad f_k(\vec{x},t) = 0 \ \forall k\in\hat{n}$

$\underbrace{\vec{F}(\vec{x},\dot{\vec{x}},t)}_{\text{vtištěné síly (akční)}} + \underbrace{\sum_{k=1}^{n}\lambda_k\nabla f_k}_{\text{vazbové síly (reakční)}} - \underbrace{\mathbb{M}\ddot{\vec{x}}}_{\text{setrvačné síly}} = 0 \quad \leftarrow$ podmínka rovnováhy sil

Zavedení setrvačné síly $\vec{I} = -\mathbb{M}\ddot{\vec{x}}$ umožňuje zobecnit princip virtuální práce ze statiky.

Rovnici opět přenásobíme $\delta\vec{x} = \delta\vec{x}^T + \delta\vec{x}^N$
kde $\delta\vec{x}^T\cdot\nabla f_k = 0 \ \forall k\in\hat{n} \quad \delta\vec{x}^N \in span_{\mathbb{R}}(\nabla f_1,\dots,\nabla f_n)$

$$\underbrace{(\vec{F}-\mathbb{M}\ddot{\vec{x}})}_{\text{efektivní síly}}\cdot\delta\vec{x}^T + \sum_{k=1}^{n}\lambda_k\underbrace{\nabla f_k\cdot\delta\vec{x}^T}_{=0} + \underbrace{(\vec{F}-\mathbb{M}\ddot{\vec{x}})^T\cdot\delta\vec{x}^N}_{=0} + \underbrace{\left[(\vec{F}-\mathbb{M}\ddot{\vec{x}})^N + \sum_{k=1}^{n}\lambda_k\nabla f_k\right]}_{=0 \text{ volbou } \lambda_k}\cdot\delta\vec{x}^N = 0 \quad \forall\delta\vec{x}^T$$

<u>Virtuální práce efektivních sil</u>

$\delta A_{ef}(\vec{x},\dot{\vec{x}},\ddot{\vec{x}},t) = (\vec{F}(\vec{x},\dot{\vec{x}},t) - \mathbb{M}\ddot{\vec{x}})\cdot\delta\vec{x}$

d'Alembertův princip (1743)

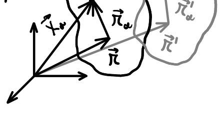
$$\boxed{\begin{gathered}\delta A_{ef} = (\vec{F}-\mathbb{M}\ddot{\vec{x}})\cdot\delta\vec{x} = 0 \quad \forall\delta\vec{x}\in\mathbb{R}^{3N} \\ f_k(\vec{x},t) = 0 \quad \forall k\in\hat{n} \qquad \delta\vec{x}\cdot\nabla f_k = 0\end{gathered}}$$

Pohyb mechanické soustavy se děje v dynamické rovnováze efektivních sil t. j. tak, že virtuální práce efektivních sil je v každém okamžiku rovna nule $\delta A_{ef}(\vec{x}(t),\dot{\vec{x}}(t),\ddot{\vec{x}}(t),t) = 0.$

Pozn: výhoda – není třeba pracovat se silama ideálních holonomních vazeb např. držících pohromadě tuhé těleso složené z N hm. bodů: posunutí jsou pouze translace a rotace $\quad \vec{x}_\alpha = \vec{r} + \vec{r}_\alpha \quad |\vec{r}_\alpha| = konst \quad \delta\vec{x}_\alpha = \underbrace{\delta\vec{r}}+\underbrace{\delta\vec{\varphi}}\times\vec{r}_\alpha$ jsou nezávislé

$$\delta A_{ef} = \sum_{\alpha=1}^{N}(\vec{F}_\alpha - m_\alpha\ddot{\vec{x}}_\alpha)\cdot\delta\vec{x}_\alpha = \sum_{\alpha=1}^{N}(\vec{F}_\alpha - m_\alpha\ddot{\vec{x}}_\alpha)\cdot(\delta\vec{r}+\delta\vec{\varphi}\times\vec{r}_\alpha) = \underbrace{\left[\sum_{\alpha=1}^{N}\vec{F}_\alpha - m_\alpha\ddot{\vec{x}}_\alpha\right]}_{=0}\cdot\delta\vec{r} + \underbrace{\left[\sum_{\alpha=1}^{N}\vec{r}_\alpha\times(\vec{F}_\alpha - m_\alpha\ddot{\vec{x}}_\alpha)\right]}_{=0}\cdot\delta\vec{\varphi} = 0 \quad \forall\delta\vec{r}\ \forall\delta\vec{\varphi}$$

$$0 = \sum_{\alpha=1}^{N}\vec{F}_\alpha - m_\alpha\ddot{\vec{x}}_\alpha = \vec{F} - \dot{\vec{P}} \qquad 0 = \sum_{\alpha=1}^{N}\vec{x}_\alpha\times(\vec{F}_\alpha - m_\alpha\ddot{\vec{x}}_\alpha) - \vec{r}\times\left(\sum_{\alpha=1}^{N}\vec{F}_\alpha - m_\alpha\ddot{\vec{x}}_\alpha\right) = \sum_{\alpha=1}^{N}\vec{x}_\alpha\times\vec{F}_\alpha - \left(\sum_{\alpha=1}^{N}\vec{x}_\alpha\times m_\alpha\dot{\vec{x}}_\alpha\right)^{\cdot} = \vec{N} - \dot{\vec{L}}$$



Pohybové rovnice (LR1D) pro neholonomní soustavu s vazbami lineárně závislými na rychlostech

vazby: r–holonomních (nezávislých) $\qquad$ p–neholonomních (nezávislých)

$f_k(\vec x,t)=0 \quad \forall k\in\hat r \qquad h(\underbrace{\nabla f_1,\dots,\nabla f_r}_{\text{L.N.}})=r \quad \forall \vec x\in M(t)$

$0=\delta f_k=\frac{\partial f_k}{\partial x_i}\delta x_i=\nabla f_k\cdot\delta\vec x$

$\sum_{i=1}^{3N} a_{li}(\vec x,t)\dot x_i + b_l(\vec x,t)=0 \quad \forall l\in\hat p \qquad \mathbb{A}=(a_{li})$ matice typu $p\times 3N$

$\mathbb{A}(\vec x,t)\dot{\vec x}+\vec b(\vec x,t)=0 \quad /\cdot dt \qquad h(\mathbb{A}(\vec x,t))=p \quad \forall\vec x\in M(t)\ \forall t$

$\mathbb{A}(\vec x,t)d\vec x+\vec b(\vec x,t)\,dt=0 \leftarrow \dot{\vec x}\,dt=d\vec x \quad / \ dt=0 \Rightarrow d\vec x\to\delta\vec x$

$\mathbb{A}(\vec x,t)\delta\vec x=0 \qquad \sum_{i=1}^{3N} a_{li}(\vec x,t)\delta x_i=0 \quad \forall l\in\hat p$

d'Alembertův princip

$$\delta A_{ef}=\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\cdot\delta x_i=(\vec F-\mathbb{M}\ddot{\vec x})\cdot\delta\vec x=0 \qquad (1)$$

$$\left.\begin{aligned}f_k(\vec x,t)&=0 \quad \forall k\in\hat r\\ \mathbb{A}(\vec x,t)\dot{\vec x}+\vec b(\vec x,t)&=0\end{aligned}\right\}(3) \qquad \forall\delta\vec x\in\mathbb{R}^{3N} \quad \left.\begin{aligned}\nabla f_k\cdot\delta\vec x&=0\\ \mathbb{A}\delta\vec x&=0\end{aligned}\right\}(2)$$

$\forall\delta\vec x\in T_{\vec x}M(t)\cap \operatorname{Ker}\mathbb{A}(\vec x,t)=\operatorname{Ker}\mathbb{B}(\vec x,t)$

$\mathbb{B}(\vec x,t)$ matice $(r+p)\times 3N$ hodnosti $r+p$

$s=3N-r$

BÚNO: regulární matice typu $(r+p)\times(r+p)$

Metoda Lagrangeových multiplikátorů:
Místo vyjadření závislých posunutí složek δx_j pomocí $s-p$ nezávislých z podmínek (2) a jejich dosazení do rovnice (1) přenásobíme rovnice (2) Lagrangeovými multiplikátory λ_j,μ_j a přičteme k rovnici (1) a nastavíme λ_j,μ_j ← funkce(t) tak abychom vynulovali koeficienty u závislých δx_j

$$\underbrace{(\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_p)}_{\text{nastavíme tak aby}}\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{s-p+1}} & \cdots & \frac{\partial f_1}{\partial x_{3N}}\\ \vdots & & \vdots & & \vdots\\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_{s-p+1}} & \cdots & \frac{\partial f_r}{\partial x_{3N}}\\ a_{1,1} & \cdots & a_{1,s-p+1} & \cdots & a_{1,3N}\\ \vdots & & \vdots & & \vdots\\ a_{p,1} & \cdots & a_{p,s-p+1} & \cdots & a_{p,3N}\end{pmatrix}\begin{pmatrix}\delta x_1\\ \vdots\\ \delta x_{s-p}\\ \delta x_{s-p+1}\\ \vdots\\ \delta x_{3N}\end{pmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}\text{nezávislé}\\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\}\text{závislé}\end{matrix}=0$$

$$\left(\overset{*}{-F_1^{ef}},\dots,\overset{\parallel}{-F_{s-p+1}^{ef}},\dots,\overset{\parallel}{-F_{3N}^{ef}}\right)$$

$$0=\sum_{i=1}^{3N}\Big(F_i-m_i\ddot x_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}+\sum_{l=1}^{p}\mu_l a_{li}\Big)\underbrace{\delta x_i}_{}=\sum_{i=1}^{s-p}\underbrace{\Big(F_i-m_i\ddot x_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}+\sum_{l=1}^{p}\mu_l a_{li}\Big)}_{}\underbrace{\delta x_i}_{}+\sum_{i=s-p+1}^{3N}\underbrace{\Big(F_i-m_i\ddot x_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}+\sum_{l=1}^{p}\mu_l a_{li}\Big)}_{}\delta x_i$$

$\delta x_1,\dots,\delta x_{3N}$ nejsou nezávislé $\qquad 0 \Leftarrow \delta x_1,\dots,\delta x_{s-p}$ jsou nezávislé $\qquad 0$ vynulujeme volbou λ_k a μ_l

Pohybové rovnice: $m_i\ddot x_i=F_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}+\sum_{l=1}^{p}\mu_l a_{li} \quad \forall i\in\widehat{3N} \qquad f_k(\vec x,t)=0 \quad \forall k\in\hat r \qquad \sum_{j=1}^{3N}a_{lj}(\vec x,t)\dot x_j+b_l(\vec x,t)=0 \quad \forall l\in\hat p$

Věta o energii: $\quad /\dot x_i \quad /\sum_{i=1}^{3N} \qquad \frac{\hat d f_k}{dt}=\frac{\partial f_k}{\partial x_i}\dot x_i+\frac{\partial f_k}{\partial t}=0 \qquad$ pro konzervativní síly $F_i\dot x_i=-\frac{\partial U(\vec x)}{\partial x_i}\dot x_i=-\frac{\hat d U(\vec x)}{dt}$

$$m_i\ddot x_i\dot x_i=F_i\dot x_i+\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial x_i}\dot x_i+\sum_{l=1}^{p}\mu_l a_{li}\dot x_i \Rightarrow \frac{\hat d T}{dt}=\vec F\cdot\dot{\vec x}-\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial t}-\sum_{l=1}^{p}\mu_l b_l \Rightarrow \frac{\hat d}{dt}(T+U)=-\sum_{k=1}^{r}\lambda_k\frac{\partial f_k}{\partial t}-\sum_{l=1}^{p}\mu_l b_l$$

Odvození LR2D pro holonomní soustavu: $\quad x_i=\hat x_i(\vec q,t) \qquad \delta x_i=\delta\hat x_i=\frac{\partial\hat x_i}{\partial q_j}\delta q_j$

$$0=\delta A_{ef}=\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\delta x_i=\sum_{i=1}^{3N}\sum_{j=1}^{s}\Big[F_i-\frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\Big)\Big]\frac{\partial\hat x_i}{\partial q_j}\delta q_j=\sum_{i=1}^{3N}\sum_{j=1}^{s}\Big[F_i\frac{\partial\hat x_i}{\partial q_j}-\frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\Big)\frac{\partial\hat x_i}{\partial q_j}\Big]\delta q_j=$$

jsou nezávislé

$$=\sum_{i=1}^{3N}\sum_{j=1}^{s}\Big[F_i\frac{\partial\hat x_i}{\partial q_j}-\frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\frac{\partial\hat x_i}{\partial q_j}\Big)+\frac{\partial T}{\partial\dot x_i}\frac{\hat d}{dt}\Big(\frac{\partial\hat x_i}{\partial q_j}\Big)\Big]\delta q_j=\sum_{i=1}^{3N}\sum_{j=1}^{s}\Big[Q_j-\frac{\hat d}{dt}\Big(\frac{\partial T}{\partial\dot x_i}\frac{\partial\hat{\dot x}_i}{\partial\dot q_j}\Big)+\frac{\partial T}{\partial\dot x_i}\Big(\frac{\partial\hat{\dot x}_i}{\partial q_j}\Big)\Big]\delta q_j=\sum_{j=1}^{s}\Big[Q_j-\frac{\hat d}{dt}\Big(\frac{\partial\hat T}{\partial\dot q_j}\Big)+\frac{\partial\hat T}{\partial q_j}\Big]\overbrace{\delta q_j}$$

$$\frac{\hat d}{dt}\Big(\frac{\partial\hat T}{\partial\dot q_j}\Big)-\frac{\partial\hat T}{\partial q_j}=Q_j \quad \forall j\in\hat s \qquad s=3N-r$$

$$Q_j=Q_j^{(nep)}+Q_j^{(pot)}=Q_j^{(nep)}+\Big[-\frac{\partial U}{\partial x_i}+\frac{\hat d}{dt}\Big(\frac{\partial U}{\partial\dot x_i}\Big)\Big]\frac{\partial\hat x_i}{\partial q_j}=Q_j^{(nep)}-\frac{\partial U}{\partial x_i}\frac{\partial\hat x_i}{\partial q_j}+\frac{\hat d}{dt}\Big(\frac{\partial U}{\partial\dot x_i}\frac{\partial\hat x_i}{\partial q_j}\Big)-\frac{\partial U}{\partial\dot x_i}\frac{\hat d}{dt}\Big(\frac{\partial\hat x_i}{\partial q_j}\Big)=$$

$$=Q_j^{(nep)}-\frac{\partial U}{\partial x_i}\frac{\partial\hat x_i}{\partial q_j}-\frac{\partial U}{\partial\dot x_i}\frac{\hat d}{dt}\Big(\frac{\partial\hat x_i}{\partial q_j}\Big)+\frac{\hat d}{dt}\Big(\frac{\partial U}{\partial\dot x_i}\frac{\partial\hat{\dot x}_i}{\partial\dot q_j}\Big)=Q_j^{(nep)}-\frac{\partial\hat U}{\partial q_j}+\frac{\hat d}{dt}\Big(\frac{\partial\hat U}{\partial\dot q_j}\Big)$$

$$\frac{\hat d}{dt}\Big(\frac{\partial\hat L}{\partial\dot q_j}\Big)-\frac{\partial\hat L}{\partial q_j}=Q_j^{(nep)} \leftarrow \hat L=\hat T-\hat U$$

Pozn: Záměnnost variace a derivace $\frac{\hat d}{dt}(\delta\vec x)=\frac{\hat d}{dt}(\vec x'-\vec x)=\dot{\vec x}'-\dot{\vec x}=\delta\dot{\vec x}=\delta\Big(\frac{\hat d\vec x}{dt}\Big)$ platí i pro funkce $\frac{\hat d}{dt}\delta f=\delta\frac{\hat d f}{dt}$

Jourdainův princip $\qquad 0=\frac{\hat d}{dt}\delta A_{ef}=\sum_{i=1}^{3N}\frac{\hat d}{dt}(F_i-m_i\ddot x_i)\underbrace{\delta x_i}_{\text{zvolíme }=0}+\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\delta\dot x_i \qquad \boxed{\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\cdot\delta_1\dot x_i=0}$

Jourdainovy variace $\quad \delta_1 t=0 \quad \delta_1 x_i=0$

Gaussův princip $\qquad 0=\sum_{i=1}^{3N}\frac{\hat d}{dt}(F_i-m_i\ddot x_i)\underbrace{\delta_1\dot x_i}_{\text{zvolíme }=0}+\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\delta_1\ddot x_i \qquad \boxed{\sum_{i=1}^{3N}(F_i-m_i\ddot x_i)\cdot\delta_2\ddot x_i=0}$

Gaussovy variace $\quad \delta_2 t=0 \quad \delta_2 x_i=0 \quad \delta_2\dot x_i=0$