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Statistical Thermodynamics of Strong Interactions at High Energies.

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CONTENTS. — **1. Introduction.** – **2. The partition function.** – **3. The self-consistency condition.** 1. Statement of the problem. 2. Exclusion of nonexponential solutions. 3. The solution of the self-consistency condition. 4. The highest temperature T_0. The model of distinguishable particles. – **4. Physical interpretation.** 1. The highest temperature T_0. 2. The other parameters. The mass spectrum. – **5. Conclusion; open questions; speculations.**

1. – Introduction.

Recently, the statistical model of Fermi [1] has been applied to large-angle elastic [2,3] and exchange [4] scattering with a rather unexpected success. Roughly, the result can be stated as follows: if one calculates with the (non-invariant) statistical model the probabilities P_j for all channels j of the reaction p+p→« channel j », then one finds for c.m. energies from 2 to 8 GeV the numerical formula

$$\left(\frac{P_0}{\sum_j P_j}\right)_{pp} = \exp[-3.30(E-2)] \qquad [E \text{ in GeV}] \tag{1}$$

(1) E. FERMI: *Progr. Theor. Phys.*, **5**, 570 (1950).
(2) G. FAST and R. HAGEDORN: *Nuovo Cimento*, **27**, 208 (1963).
(3) G. FAST, R. HAGEDORN and L. W. JONES: *Nuovo Cimento*, **27**, 856 (1963).
(4) R. HAGEDORN: *Nuovo Cimento*, **35**, 216 (1965).

[our units are: $\hbar = c = k$ (Boltzmann's constant) $= 1$] where P_0 denotes the elastic channel. From this $(d\sigma_{el}/d\omega)_{90°}$ was calculated in ref. [2-4].

The natural question arose whether this numerical results could be understood. A number of authors considered the asymptotic behaviour of sums over phase-space integrals [5-7] or treated the question by thermodynamical methods [8,9].

The analytic treatment [5-7] resulted always in

$$\frac{P_0}{\sum_j P_j} \sim \exp[-aE^\alpha] \qquad \text{for } E \to \infty, \tag{2}$$

where $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1$ according to various assumptions about details.

It turned out that $\alpha = 1$ is impossible under the usual assumption that the particles are indistinguishable. If one omits, however, the factor $1/n!$ in front of the phase-space integrals then $\alpha = 1$ results. This was first pointed out by AUBERSON and ESCOUBÈS [5] for the noninvariant, and later by VANDERMEULEN [6] for the invariant phase space.

Omitting the factor $1/n!$ corresponds to consider the particles as being distinguishable. This is not altogether senseless, since in reality there are so many different interacting particles (henceforward: hadrons) involved in the high-energy processes that the average occupation number in the numerical calculations leading to (1) remains below ≈ 1 (where, of course, the different charge and spin states have to be counted separately). In this situation the particles behave *as if* they were distinguishable. Responsible for this—and presumably for the actual behaviour of strong interactions at high energies—is the large number of possible particle states. Indeed, the number of particles and resonances seems to grow very fast as a function of their mass [10]. Since only a

(5) G. AUBERSON and B. ESCOUBÈS: *Production of light bosons by central collisions at very high energy*, CERN preprint 9323/TH 460, 4 August 1964 (to be published in *Nuovo Cimento*, 1965).
(6) J. VANDERMEULEN: *Elastic scattering in high-energy proton-proton central collisions*, preprint of Départment de Physique Mathématique, Université de Liège, July 1964 (to be published).
(7) H. SATZ: *The high-energy limit of the statistical model*, August 1964 (Summer Institute for Theoretical Physics, University of Wisconsin, Madison, Wisc.; now at DESY, Hamburg).
(8) A. BIAŁAS and V. F. WEISSKOPF: *Statistical theory of elastic proton-proton scattering at large angles*, CERN preprint 9236/TH. 461, 5 August 1964 (to be published in *Nuovo Cimento*, 1965).
(9) G. COCCONI: *Nuovo Cimento*, **33**, 643 (1964). COCCONI says: «... corresponds to the case of a system in which, for E increasing, the number of possible kinds of particles increases so as to keep the energy per particle, and hence the temperature, constant ». Our whole theory is implicitly contained in this remark.
(10) A. H. ROSENFELD, A. BARBARO-GALTIERI, W. H. BARKAS, P. L. BASTIEN, J. KIRZ and M. ROOS: *Rev. Mod. Phys.*, **36**, 977 (1964) and UCRL 8030.

few of them are stable against strong decay and in principle no criterion exists to distinguish between a resonance (which decays strongly) and the so-called fire-balls (which seem to be observed at very high energies) we shall assume that the fire-balls and the low-lying resonances are basically one and the same thing, namely, excited hadrons. Thus, we may speak of a mass spectrum of hadrons, which has at low masses a few discrete lines; between and above these some rather sharp resonances; for larger mass values more and more resonances appear which become gradually broader until, finally, there are so many that the width becomes comparable to their spacing and a continuous spectrum of fire-balls results. Strictly speaking the fire-balls, which we presently have in mind and which are counted by a mass spectrum, are not exactly the same as those actually observed: the former ones are states with well-defined quantum numbers (except the mass itself) whereas the latter ones can be mixtures of states within a mass interval Δm.

A high-energy collision will then be visualized in this paper as follows: in the first instant in a certain volume V_0 a thermodynamical « equilibrium » (*) is established which will be described by statistical thermodynamics of an unlimited and undetermined number of more or less excited hadrons, which then leave the region of interaction and decay (strongly) through a number of steps into « stable » forms K, π, $\mathcal{N}$, Y, etc.

The essential idea is now the following one: the thermodynamical system consisting of more or less excited hadrons is itself nothing else than a highly excited hadron (because in the sense of our above statement we have no way to distinguish between a resonance, a fire-ball and our thermodynamical system—except that they differ in the degree of excitation). This leads then to a self-consistency problem:

1) let $\varrho(m)\mathrm{d}m$ denote the mass spectrum of hadrons, *i.e.*, the number of (more or less) excited hadrons with mass between m and $m+\mathrm{d}m$

2) and let $\sigma(E)\mathrm{d}E$ denote the number of states between E and $E+\mathrm{d}E$ of our thermodynamical system. (E is always the total energy including the rest masses of the particles.)

Then, if there is no essential difference between excited hadrons and our thermodynamical system, that is, if excited hadrons can—for very high excitation ($m\to\infty$)—be described themselves by the same formalism, then the two functions $\varrho(x)$ and $\sigma(x)$ should asympotically approach each other for $x\to\infty$.

This self-consistency condition (asymptotic bootstrap) may or may not be accepted as a statement about nature, it may or it may not be considered

(*) The word equilibrium will turn out to have a meaning which is somewhat different from the usual one and which is better expressed replacing « equilibrium » by « constant temperature ». See footnote on p. 163.

intuitively obvious—we shall, in this paper, adopt it as a working hypothesis and find out its consequences.

It will be shown that the functions $\varrho(x)$ and $\sigma(x)$ *must* grow exponentially. An immediate consequence is the existence of a highest possible temperature T_0, which then should govern practically all high-energy phenomena in which hadrons take part. Once this temperature is (nearly) reached—in other words: for collisions of sufficiently large total energy and momentum transfer—the reaction will be described by thermodynamical and conservation laws. Neither the details of the interaction nor the structure of the interacting hadrons will manifest themselves anymore when both energy and momentum transfer become large.

The existence of a highest temperature can be understood by observing that the density $\sigma(E)$ of states of a system grows already very fast (almost exponentially) if only one kind of particles is available. This growth expresses the fact that by increasing the kinetic energies of the particles so many new energy levels in the box V_0 become available. But if there is the possibility to create ever new kinds of particles and if the density of particle states (other than kinetic energy) grows very fast, then the system will answer an increase of energy by increasing both the kinetic energy ($\sim$ temperature) and the number of kinds of particles. It then happens that for an exponential growth of $\varrho(m)$ the system uses up the energy to increase the temperature and the number of particles only up to some temperature $\approx T_0$; but when T_0 is approached it becomes easier to create new particles than to increase the temperature; $\varrho(m)$ offers more possibilities than the increase of the number of momentum states does (9). Example: for an ideal gas (with fixed particle number) the temperature T grows proportionally to the energy E; for the light quantum gas, where particles can be created, the temperature increases only as $T\sim E^{\frac{1}{4}}$. In our case, where the number of available types of particles itself increases with the energy, $T\to$ constant.

It should be noted that we do not make the a priori assumption that the particles be distinguishable; we shall start from the proper quantum statistics where equal particles are indistinguishable. It turns out, however, that, after imposing the self-consistency condition on $\varrho(m)$ and $\sigma(E)$, the system behaves very similarly to a model system in which all particles are distinguishable.

In Sect. 2 we derive the basic statistical formula; Sect. 3 discusses the self-consistency condition; Sect. 4 gives the physical interpretation and Sect. 5 summarizes the results and contains some speculations and open problems.

2. – The partition function.

We deal with a system of particles enclosed in a volume V_0 and kept at constant temperature T (canonical ensemble). The number of particles of each kind is not limited. We label by α the possible momenta in the cubical box V_0 and by β the kinds of particles.

Then the partition function becomes

$$Z(V_0, T) = \sum_{(\nu)} \exp\left[-\frac{1}{T}\sum_{\alpha\beta} \varepsilon_{\alpha\beta}\nu_{\alpha\beta}\right] - 1, \qquad \varepsilon_{\alpha\beta} \equiv \sqrt{p_\alpha^2 + m_\beta^2}. \tag{3}$$

The sum (ν) goes over all sets (matrices) with matrix elements $\nu_{\alpha\beta} = 0, 1, 2, ..., \infty$; $\nu_{\alpha\beta}$ is the number of particles of the kind β having momentum $\boldsymbol{p}_\alpha$. Thus, one particular matrix ν describes fully one quantum state of the system. The contribution 1 of the ground state (no particle present: $\nu_{\alpha\beta} \equiv 0$) has been subtracted. Since the relevant physical quantites are obtained from logarithmic derivatives of $Z(V_0, T)$ this is of no other consequence than to ensure that $Z(V_0, T)$ can be written as the Laplace transform of a positive function, namely the density of states $\sigma(E)$, which then has no δ-function behaviour at $E = 0$.

We write with $x_{\alpha\beta} \equiv \exp[-\varepsilon_{\alpha\beta}/T]$

$$1 + Z(V_0, T) \equiv \sum_{(\nu)} \prod_{\alpha\beta} x_{\alpha\beta}{}^{\nu_{\alpha\beta}} = \left(\sum_{\nu_{11}} x_{11}{}^{\nu_{11}}\right)\left(\sum_{\nu_{12}} x_{12}{}^{\nu_{12}}\right) \dots \left(\sum_{\nu_{\alpha\beta}} x_{\alpha\beta}{}^{\nu_{\alpha\beta}}\right) \dots .$$

Let us distinguish bosons (label β) and fermions (label φ). Then $\nu_{\alpha\beta} = 0, 1, 2, ..., \infty$ whereas $\nu_{\alpha\varphi} = 0$ or 1 only. Thus

$$Z(V_0, T) = \prod_{\alpha\beta} \frac{1}{1 - x_{\alpha\beta}} \prod_{\alpha\varphi}(1 + x_{\alpha\varphi}) - 1,$$

$$\log[1 + Z(V_0, T)] = -\sum_{\alpha\beta} \log(1 - x_{\alpha\beta}) + \sum_{\alpha\varphi} \log(1 + x_{\alpha\varphi}).$$

We replace (without any consequence for later conclusions)

$$\begin{cases} \displaystyle\sum_\alpha \to \int_0^\infty \frac{V_0 \cdot 4\pi p^2}{h^3}[\dots]\,\mathrm{d}p = \frac{V_0}{2\pi^2}\int_0^\infty p^2[\dots]\,\mathrm{d}p, \\ \displaystyle\sum_{\beta,\varphi} \to \int_0^\infty \varrho_{\mathrm{B,F}}(m)[\dots]\,\mathrm{d}m, \end{cases} \tag{4}$$

and obtain

$$\log[1+Z(V_0, T)] = \frac{V_0}{2\pi^2}\int_0^\infty p^2\,\mathrm{d}p\left[\int_0^\infty \mathrm{d}m[\varrho_{\mathrm{F}}(m)\log(1 + x_{pm}) - \varrho_{\mathrm{B}}(m)\log(1 - x_{pm})]\right].$$

Expanding the logarithms yields

$$\begin{cases} \displaystyle\log[1 + Z(V_0, T)] = \frac{V_0}{2\pi^2}\sum_1^\infty \frac{1}{n}\int_0^\infty \mathrm{d}p\,\mathrm{d}m[[\varrho_{\mathrm{B}}(m) - (-)^n\varrho_{\mathrm{F}}(m)]\cdot p^2 \cdot x_{pm}^n], \\ \displaystyle x_{pm}^n = \exp\left[-\frac{n}{T}\sqrt{p^2 + m^2}\right] < 1, \end{cases} \tag{5}$$

(no hadrons with $m = 0$ exist).

For the combined mass distributions we write

$$\varrho_{\mathrm{B}}(m) - (-)^n\varrho_{\mathrm{F}}(m) \equiv \varrho(m; n) \equiv \begin{cases} \varrho(m) & \text{for } n \text{ odd}, \\ \Delta\varrho(m) & \text{for } n \text{ even}. \end{cases} \tag{6}$$

The integral over p can be solved (11)

$$\int_0^\infty \mathrm{d}p\,p^2 \exp\left[-\frac{n}{T}\sqrt{p^2 + m^2}\right] = -m^3\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{K_1(y)}{y}\right] = \frac{m^3}{y}K_2(y), \qquad y = \frac{nm}{T}, \tag{7}$$

$K_{1,2}(y)$ are modified Hankel functions. Finally

$$\log[1 + Z(V_0, T)] = \frac{V_0 T}{2\pi^2}\sum_{n=1}^\infty \frac{1}{n^2}\int_0^\infty \varrho(m; n)\,m^2 K_2\left(\frac{nm}{T}\right)\mathrm{d}m. \tag{8}$$

Indeed, for $T \to 0$ the r.h.s. vanishes, hence $Z(V_0, T \to 0) \to 0$.

3. – The self-consistency condition.

3·1. *Statement of the problem.* – Now we impose the condition (explained in the Introduction) that in the limit $m \to \infty$ the highly excited hadrons (fireballs) should be themselves describable by statistical thermodynamics, *i.e.*, by eq. (8) and its consequences.

To fomulate this condition, we observe that

$$Z(V_0, T) = \exp\left[\frac{V_0 T}{2\pi^2}\sum_1^\infty \frac{1}{n^2}\int_0^\infty \varrho(m; n)\,m^2 K_2\left(\frac{nm}{T}\right)\mathrm{d}m\right] - 1 \tag{9}$$

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(11) For the following see, *e.g.*, I. M. Ryshik and I. S. Gradstein: *Tables of Series, Products and Integrals*, chap. 6.4 (Berlin, 1957) or any book on Bessel functions.

can be written in another form

$$Z(V_0, T) = \int_0^\infty \sigma(E) \exp\left[-\frac{E}{T}\right] dE\,, \tag{10}$$

where $\sigma(E)dE$ counts the number of states between E and $E + dE$. That such a representation is possible follows from the physical meaning of our equations [compare (3)]. It is, however, also a consequence of rather general theorems on Laplace transformations. As some of our following arguments will be based on such theorems, we shall adapt our notation a little by introducing the variable

$$s = \frac{1}{T}, \tag{11}$$

without changing the function symbols. Then

$$Z(s) = \exp\left[\frac{V_0}{2\pi^2}\cdot\frac{1}{s}\cdot\sum_1^\infty \frac{1}{n^2}\int_0^\infty m^2 K_2(nsm)\,\varrho(m;\,n)\,dm\right] - 1 \equiv \int_0^\infty \exp[-sE]\sigma(E)\,dE\,. \tag{12}$$

We express our self-consistency condition in the rather weak form

$$\log\varrho(x) \to \log\sigma(x) \qquad \text{for } x\to\infty\,. \tag{13}$$

We shall be able to show that (13) can be achieved. From the physical point of view (13) seems rather natural, as $\log\sigma(E)$ is essentially the entropy of our system. Since we are trying a thermodynamical description in which not only the conservation laws but also the details of the interaction are disregarded, we should perhaps not require more than the asymptotic equality of the entropies. We should, indeed, rather expect that $\varrho(x)$ and $\sigma(x)$ differ by some algebraic factor in x. The reason is that $\sigma(E)$ counts all the states of the system enclosed in the box V_0, among them, for instance, also states of very large total angular momentum (= collective motion) which we would not like to interpret to be « fire-balls » and which therefore should not be counted in $\varrho(m)$, but which cannot be excluded from Z unless we build in explicitly some restrictions (conservation laws, centrality condition (4)). The « particles » counted by $\varrho(m)$ should, however, be objects which are not simply n-particle states with a total invariant mass m, but which still show some properties which are reminiscent of what one calls a « compound system » in nuclear physics and « fire-balls » in cosmic-ray physics. A qualitative way to classify such objects would be to say that they are those which could have been generated by a *central* collision of two appropriate « elementary » particles (low-lying states of our family of hadrons). The total c.m. energy of such a generating collision should be just equal to m, and if m is large then it comes mainly from the kinetic energy of the incoming particles. The condition that a collision produces a compound system, *i.e.*, has been central, can be stated in terms of a simple model (4) by saying that the total angular momentum of such a compound system must not exceed $l\approx 5$ to 7, independent of the primary energy, that is, independent of its total mass. This can also be formulated by saying that of all the states which can be reached from the initial one, only the fraction $1/\gamma^2 \sim 1/m^2$ can be considered to be of the compound type [see ref. (4), eq. (5); γ = Lorentz factor of the c.m.; for two equal initial particles with mass m_0 one has $\gamma = m/2m_0$]. We thus might expect that $\varrho(x)$—counting only such compound states—could be smaller than $\sigma(x)$ by a factor of the order $1/x^2$ —in any case, we should tolerate factors of this kind. In the weak form (13) of our self-consistency condition such factors are permitted (we anticipate that ϱ and σ must grow faster than any power).

3'2. *Exclusion of nonexponential solutions.* – We consider now the mathematical problem given by eqs. (12) and (13). In what follows we shall quote theorems on Laplace transformations from DOETSCH (12); our notation is explained in Appendix 2.

The modified Hankel function $K_2(x)$ has the properties (11) [see (A4.10)]

$$\left\{\begin{aligned} K_2(x) &\xrightarrow[x\to\infty]{} \sqrt{\frac{\pi}{2x}}\exp[-x]\left(1+\frac{\theta}{x}+O\left(\frac{1}{x^2}\right)\right), \qquad |\theta|<1\,,\\ K_2(x) &\xrightarrow[x\to 0]{} \frac{2}{x^2}+O\left(x^2\log\frac{x}{2}\right). \end{aligned}\right. \tag{14}$$

Thus the integrand in (12) becomes

$$\left\{\begin{aligned} \varrho(m;\,n)m^2K_2(nms) &\to \frac{2}{n^2s^2}\,\varrho(m;\,n) \qquad &\text{at } m\to 0\,,\\ \varrho(m;\,n)m^2K_2(nms) &\to \sqrt{\frac{\pi}{2}}\,\frac{m^{\frac{3}{2}}\varrho(m;\,n)}{\sqrt{ns}}\exp[-mns] \qquad &\text{at } m\to\infty\,. \end{aligned}\right. \tag{15}$$

(12) G. DOETSCH: *Handbuch der Laplace-Transformation*, Bände I, II, III (Basel 1950/55). We quote as follows: Doetsch, Satz 4, I 11.3 means the theorem n. 4 in Vol. I, Chap. 11, Sect. 3.

No problem arises at the lower limit of the integral. At the upper limit

$$\int_0^\infty m^2 K_2(nms)\,\varrho(m;n)\,\mathrm{d}m < \infty \qquad \text{for } \operatorname{Re} s > s_0 \tag{16}$$

if

$$\varrho(m;n) = O(\exp[s_0 m])\,.$$

In particular, all integrals ($n = 1, 2, 3, \dots \infty$) converge if the first ($n = 1$) does so. In Appendix 1 we show that

(17)
1) if the integral with $n = 1$ converges, the sum in (12) converges;
2) if the integral with $n = 1$ does not converge, but the one with $n = 2$ converges, then the sum from $n = 2$ to ∞ converges.

This will enable us later, when we discuss divergences for $s \to s_0^+$, to disregard all terms of the sum except the first one.

Next we observe by going back to eq. (3)—replacing there $1/T$ by s—that $Z(s)$ is completely monotonic for $s > 0$, *i.e.*,

$$(-)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} Z(s) \geqslant 0 \qquad \text{for } s > 0\,. \tag{18}$$

This property is not so obious in the form (12), since $\Delta\varrho(m)$ might be negative. According to Bernstein's theorem [13] every completely monotonic function can be represented by a Stieltjes-Laplace integral [the second line of eq. (12)] (*) with $\sigma(E) \geqslant 0$. This is physically obvious but mathematically provides a firm basis to all our considerations—whatever $\varrho(m) \geqslant 0$ may be.

Assume now we start with some knowledge of $\varrho(m)$, say with $\varrho(m) = 3\cdot\delta(m - m_\pi)$, and neglect the rest. In that case we find for $s \to 0^+$

$$Z(s) \to \exp\left[\frac{\text{const}}{s^3}\right], \tag{19}$$

a strongly diverging function. One can calculate the corresponding $\sigma(E)$ and find it growing like $\exp[a\cdot E^{3/4}]$. The self-consistency condition implies then that $\varrho(m) = 3\delta(m - m_\pi)$ is insufficient and that $\varrho(m)$ has also to grow like

(13) See *e.g.* D. V. WIDDER: *The Laplace Transform* (Princeton, 1964).
(*) As $\sigma(E)$ and $\varrho(m)$ may contain δ-functions, the integrals are in fact of the Stieltjes type.

$\exp[a\cdot m^{3/4}]$. With such a $\varrho(m)$, however, $Z(s)$ would diverge even faster than indicated by eq. (19); $\sigma(E)$ would in turn also grow faster than $\exp[a\cdot E^{3/4}]$ and consequently $\varrho(m)$ too. All this originates in the strong $\exp[1/s^3]$ divergence of $Z(s)$ already for the most modest $\varrho(m)$. It will be shown that continuing as above we never would satisfy the self-consistency condition (Appendix 3).

The situation is completely changed if we admit exponentially growing $\sigma(x)$ and $\varrho(x)$ of the type $\sim \exp[s_0 x]$ with $s_0 > 0$. Then, all integrals will converge for $\operatorname{Re} s > s_0$ where s_0 is necessarily the same in σ and ϱ. In that case the term corresponding to $1/s^3$ in eq. (19) can, for $s \to s_0^+$, be replaced by $1/s_0^3$ and the worst divergence disappears. Now, only the divergence for $s \to s_0^+$ of the mass integral in the exponent of eq. (12) is relevant and this divergence can be manipulated (even removed) such that the self-consistency requirement is met with.

Again, the situation is completely changed if we admit $\varrho(m)$ and $\sigma(E)$ growing faster than exponentially: all integrals diverge for all $s < \infty$ and no thermodynamics is possible. Thus such functions are inadmissible.

Since exponential solutions will be shown to be possible, since furthermore solutions growing faster than exponentially are excluded, and since finally for any assumed $\varrho(m)$ growing less than exponentially the resulting $\sigma(E)$ is always trapped between $\varrho(m)$ and exponential growth:

$$\varrho(x) = o(\sigma(x)) = o(\exp[ax]) \qquad \text{for all } a > 0\,; \tag{20}$$

since this is, so we are forced into the exponential behaviour of $\varrho(m)$ and $\sigma(E)$ as the only possible one. This implies the existence of some $s_0 > 0$ or, in other words, of a highest temperature $T_0 = 1/s_0$. We have made it sufficiently plausible that a nonexponential solution cannot exist. The proof is found in Appendix 3.

In the following Section we shall show that the self-consistency condition can be fulfilled by functions of the exponential type.

3·3. *The solution of the self-consistency condition.* – We try the following ansatz for ϱ and σ (asymptotic form)

$$\begin{cases} m^{3/2}\varrho(m) \to a m^\alpha \exp[s_0 m] + O(\exp[s_1 m]) \\ \sigma(E) \quad \to b E^\beta \exp[s_0 E] + O(\exp[s_1 E]) \end{cases} \tag{21}$$

(where, if necessary, polynomials may replace the single powers). We suppose that $s_1 \leqslant s_0 - \varepsilon$ with $\varepsilon > 0$ so that we really have an isolated singularity. In fact, this is more general than it seems, because it includes also certain superpositions of such functions with various $s \leqslant s_0$, namely functions of the type

$$\varrho(m) \to \int_0^{s_0} f(m, x) \exp[xm] \, dx ,$$

which can formally be integrated by changing variables

$$\varrho(m) \to \frac{\exp[s_0 m]}{m} \int_0^{ms_0} f\left(m, s_0 - \frac{y}{m}\right) \exp[-y] \, dy \xrightarrow[m\to\infty]{} \exp[s_0 m] \frac{f(m, s_0)}{m} .$$

Thus, we only have to require that $f(m, s_0)$ has an asymptotic expansion bounded by a power of m; in that case, we are back to the ansatz (21). Of course, we limit by the ansatz (21) seriously the class of functions we admit for trial, but presently we only want to show that there *are* solutions of the type (21) without aiming to prove that there might not also be solutions of the exponential type where s_0 is not an isolated singularity of $Z(s)$. (We suppose, of course, that s_0 is the singularity with the largest real part, so that $Z(s)$ is holomorphic in the half-plane $\operatorname{Re} s > s_0$.)

As s_0 is a singularity of $Z(s)$, the value of s_0 must be the same in ϱ and in σ. Our supposition (21) of the asymptotic form of ϱ and σ makes it possible to use an « Abelian theorem » [Doetsch, Satz 1, II 4.2] which we quote here in a form adapted to our situation:

If $F(x)$ has for $x \to \infty$ the asymptotic form

22a) $$F(x) = ax^{\gamma} \exp[s_0 x] + O(\exp[s_1 x])$$

(with arbitrary real a, γ, s_0, s_1 such that $s_1 \leqslant s_0 - \varepsilon$, $\varepsilon > 0$ then the Laplace transform $f(s)$ of $F(t)$ exists (trivially) and is holomorphic in the half-plane $\operatorname{Re} s > s_0$, but it can be analytically continued into the half-plane $\operatorname{Re} s > s_1$ except for a singularity at s_0 with the leading part

(22b) $$\frac{a\Gamma(\gamma+1)}{(s-s_0)^{\gamma+1}} \qquad \text{for } \gamma \neq -1, -2, \ldots,$$

(22c) $$\frac{a(-1)^p}{(p-1)!} \cdot (s-s_0)^{p-1} \cdot \log(s-s_0) \qquad \text{for } \gamma = -p = -1, -2, \ldots .$$

It is, of course, clear what we must do: for ϱ we take such a power that in the exponent of eq. (12) we obtain $\sim \log[1/(s-s_0)]$ according to (22c)—which is the only possibility to get after exponentiation something which again is of the form (22b-c)—and for σ such a power that we obtain $\sim 1/(s-s_0)$. Then the two

functions $\varrho(x)$ and $\sigma(x)$ differ asymptotically only by a power of x and thus fulfil our self-consistency requirement. For details see Appendix 4.

Roughly then, it goes as follows: we first adopt the simpler notation introduced in Appendix 3, (A3.1)–(A3.4), and write

(23) $$\begin{cases} Z(s) = \exp\left[\dfrac{V_0}{(2\pi)^{\frac{3}{2}}} \cdot \dfrac{1}{s^3} \cdot \displaystyle\int_0^{\infty} m^{\frac{3}{2}} \varrho(m) Q(sm) \, dm\right] - 1 , \\ Z(s) = \displaystyle\int_0^{\infty} \sigma(E) \exp[-sE] \, dE . \end{cases}$$

We here only need the asymptotic behaviour of $Q(sm)$ for fixed $s > 0$ and $m \to \infty$

(24) $$Q(sm) \to s^{\frac{3}{2}} \exp[-sm] \qquad \text{for } sm \to \infty .$$

This is sufficient, since for $s \to s_0^+$ any other term of the sum occuring in the exponent (compare Appendix 3, eqs. (A3.1) and (A3.2)) will bear a factor $\exp[nsm]$ with $n \geqslant 2$ and thus not lead to a singularity at s_0 but at $s_0/n < s_0$. Furthermore, we have proved in Appendix 1 that the rest of the sum converges. Thus every contribution other than that coming from (24) will be holomorphic at s_0 and need not be considered here, as it can be replaced by its value at s_0 when $s \to s_0^+$.

Now, with (24) we write

(25) $$\int_0^{\infty} m^{\frac{3}{2}} \varrho(m) Q(sm) \, dm = \int_0^{M} m^{\frac{3}{2}} \varrho(m) Q(sm) \, dm + s^{\frac{3}{2}} \int_M^{\infty} m^{\frac{3}{2}} \varrho(m) \exp[-sm] dm ,$$

where M has to be chosen sufficiently large. The first integral gives a function holomorphic at s_0 and the second one can be evaluated for $s \to s_0^+$ by the theorem (22a-c) provided $m^{\frac{3}{2}} \varrho(m)$ is of the form (21). As we show in Appendix 4, a term of the form

(26) $$m^{\frac{3}{2}} \varrho(m) \to \frac{a}{m} \exp[s_0 m]$$

gives, indeed, the required logarithmic behaviour. We may then evaluate the integral (25) to obtain

(27) $$\int_0^{\infty} m^{\frac{3}{2}} \varrho(m) Q(sm) \, dm = F(s, M) + as^{\frac{3}{2}} \int_M^{\infty} \frac{dm}{m} \exp[-(s-s_0)m] .$$

Here the first integral $F(s, M)$ can be numerically calculated if the mass spectrum up to the mass M is known experimentally. Of course, the mass M is defined to be so large that for $m > M$ the asymptotic forms (24) and (26) hold. In that case, the r.h.s. of (27) is in fact independent of M.

The second integral is the well-known exponential integral [11]

$$\int\limits_M^\infty \frac{\mathrm{d}m}{m}\exp[-(s-s_0)m] = -C + \log\frac{1}{M(s-s_0)} - \sum_{k=1}^{\infty}\frac{(-)^k[M(s-s_0)]^k}{k!\cdot k}, \tag{28}$$

$$C = \text{Euler's constant} = 0.5772\ldots .$$

For $s\to s_0$ we drop the sum. Inserting everything into (23) yelds

$$Z(s)\xrightarrow[s\to s_0^+]{}\exp\left[\frac{V_0}{(2\pi)^{\frac{3}{2}}}\left[\frac{F(s_0, M)}{s_0^3} + \frac{a}{s_0^{\frac{3}{2}}}\left(\log\frac{1}{M(s-s_0)} - C\right)\right]\right] = \tag{29}$$

$$= \left(\frac{1}{s-s_0}\right)^{aV_0/(2\pi s_0)^{\frac{3}{2}}}\cdot\exp\left[\frac{aV_0}{(2\pi s_0)^{\frac{3}{2}}}\left[\frac{F(s_0, M)}{as_0^{\frac{3}{2}}} - C - \log M\right]\right],$$

where in the second form the argument of exp is constant. $\sigma(E)$ must then have the asymptotic form

$$\sigma(E)\xrightarrow[E\to\infty]{} bE^{\alpha-1}\exp[s_0 E], \qquad \alpha\neq 2, 3, 4, \ldots, \tag{30}$$

which will give [see (22a-c)]

$$\int\limits_0^\infty \sigma(E)\exp[-sE_0]\mathrm{d}E\xrightarrow[s\to s_0^+]{}\left(\frac{1}{s-s_0}\right)^{\alpha}\cdot b\cdot\Gamma(\alpha). \tag{31}$$

As (29) and (31) should become asymptotically equal, we require

$$\begin{cases} \alpha = \dfrac{aV_0}{(2\pi s_0)^{\frac{3}{2}}}, \\ b\Gamma(\alpha) = \exp\left[\dfrac{aV_0}{(2\pi s_0)^{\frac{3}{2}}}\left[\dfrac{F(s_0, M)}{as_0^{\frac{3}{2}}} - C - \log M\right]\right] \end{cases} \tag{32}$$

What is the situation now? We have shown that a solution of the exponential type is possible and that then $m^{\frac{5}{2}}\varrho(m)\to(a/m)\exp[s_0 m]$ is the asymptotic behaviour of the mass spectrum. There are, however, several parameters left open, namely:

- $s_0 = 1/T_0$ the inverse of the « highest temperature »;
- a and b constants in the mass spectrum and $\sigma(E)$;
- α the exponent in $\sigma(E)$;
- V_0 the volume of our box.

There is finally M; but if our theory is correct and if M is chosen sufficiently large, then nothing should depend on M [see (27)].

We thus have five parameters with two equations (32) which leaves us with three free parameters describing the *asymptotic behaviour* of the mass spectrum and of high-energy collisions. The function $F(s_0, M)$ being an integral over the low-lying mass values up to M has no direct relation to the asymptotic behaviour and will thus be considered to be a given information and not as a set of free parameters. It is clear that this function which is equivalent to the mass spectrum for $m < M$ cannot be derived from the present theory. The actual value of $F(s_0, M)$ is indeed not relevant, as it obiously drops out when one calculates physical quantities like $\bar{E}$, $\bar{N}$ (the mean particle number) etc., from the logarithmic derivative of $Z(s)$.

In Sect. 4 we shall try to reduce the number of parameters and obtain limitations on their possible values.

3·4. *The highest temperature T_0. The model of distinguishable particles.* –

Our result (29) is, if written with $T\equiv 1/s$,

$$Z(T)\xrightarrow[T\to T_0^-]{}\text{const}\cdot\left(\frac{1}{T_0 - T}\right)^{\alpha} \tag{33}$$

The expectation value of the energy becomes

$$\bar{E}(T) = -\frac{\mathrm{d}\log Z(s)}{\mathrm{d}s}\xrightarrow[T\to T_0^-]{}\alpha\cdot\frac{T_0^2}{T_0 - T}. \tag{34}$$

It diverges (simple pole) for $T\to T_0^-$; the relative fluctuations are

$$\frac{\overline{E^2} - \bar{E}^2}{\bar{E}^2} = -\frac{1}{\bar{E}^2}\frac{\mathrm{d}\bar{E}}{\mathrm{d}s} = \frac{1}{\alpha} \tag{35}$$

(of order one as we shall see). That is, when $T\to T_0^-$, the fluctuations become as large as the energy itself. This behaviour implies that we not only can calculate $\bar{E}$ as a function of T as usual, but also can conversely say that, if a system of energy E is given, a temperature $T(E)$ belongs to it which becomes better and better defined—and equal to T_0^-—when E grows larger and larger. This is easily understood by imagining the following « Gedankenexperiment ». Somebody keeps for us a system (fire-ball) in a temperature bath T until equilibrium is established. Then he takes it out and isolates it. We are allowed to measure its energy E. Knowing E, but not T, what can we say about what T might have been?

We may invert eq. (34) replacing $\bar{E}$ by the actual value E which we found and then say $T(E)$ was most likely the temperature of the bath. Using the fact that for $E \to \infty$ the temperature $T \to T_0^-$ and the relative fluctuations (35) $\to 1/\alpha$ [the absolute fluctuations $\to \bar{E}(T)$], we see that our guess will become safer and safer the larger the actual E was. Indeed: if E was very large, then the temperature T cannot have been much lower than T_0; because, if it would have had, the probability for having found the actual large E would have been practically zero because the most likely energy values lie roughly between 0 and $2\bar{E}(T)$ according to (35).

Hence, to every given energy E one can assign a temperature $T(E)$—the inverse function of $\bar{E}(T)$—which with a certain « probability » is the one which, in a hypothetical temperature bath, the system has had before. The « probability » (*) that the actual temperature was indeed T tends to one and the temperature T tends to T_0^- when the energy goes to infinity. We may then forget about the canonical ensemble and temperature bath and simply state:

For the physical systems described by our present theory we may assign to any given energy E a temperature $T(E)$, which is not « sharp » but which becomes sharp and tends to T_0^- when $E \to \infty$. In this way we associate a well-defined temperature T_0 to all strong interactions at sufficiently high energy and momentum transfer—independent of the actual number of particles (for this latter number a probability distribution can be calculated) (**).

It is remarkable that a very similar behaviour is found when one constructs a model of distinguishable particles. The partition function has there a simple pole (no branch point) at the point T_0. Indeed, if $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i, \ldots$ are the energy levels then, with occupation numbers $n_1, n_2, \ldots, n_i, \ldots$, there will be

$$\frac{N!}{n_1!\, n_2! \ldots n_i! \ldots} \text{ states of energy } E = \sum_i n_i \varepsilon_i\,, \qquad N = \sum_i n_i\,.$$

In this case, if particles can be freely created

$$Z = \sum_{N=0}^{\infty} \sum_{\sum n_i = N} \frac{N!}{n_1!\, n_2! \ldots} \exp\left[-\frac{1}{T}\sum n_i \varepsilon_i\right] = \sum_{N=0}^{\infty}\left[\sum_i \exp\left[-\frac{\varepsilon_i}{T}\right]\right]^N \tag{36}$$

Neglecting masses and replacing $\sum_i$ by

$$\frac{4\pi V_0}{h^3}\int_0^\infty p^2 \exp\left[-\frac{p}{T}\right] \mathrm{d}p = \frac{V_0 T^3}{\pi^2}$$

(*) This is what one frequently calls an « inverse probability ».

(**) Preliminary calculations show that the number of particles is distributed approximately according to a Poisson law with $\bar{N}$ proportional to $\log E$. N is however not the number of *final* particles; it is the number of « fire-balls » leaving V_0 and decaying later. E is the total energy only in a central collision, otherwise $E < E_{\text{c.m.}}$.

gives

$$Z = \frac{1}{1 - \sum_i \exp[-\varepsilon_i/T]} = \frac{1}{1 - V_0 T^3/\pi^2} = \frac{T_0^3}{T_0^3 - T^3}\,, \qquad T_0^3 \equiv \frac{\pi^2}{V_0}\,, \tag{37}$$

which for $T \to T_0^-$ becomes

$$Z(T) \to \frac{\text{const}}{T_0 - T}\,. \tag{38}$$

We see that our theory would give the same behaviour if in (33)

$$\alpha = \frac{a V_0 T_0^{\frac{3}{2}}}{(2\pi)^{\frac{3}{2}}} = 1 \qquad \text{and} \qquad a = \sqrt{\frac{8\pi}{V_0}}\,. \tag{39}$$

Taking the « natural interaction volume »

$$V_0 \approx \frac{4\pi}{3}\left(\frac{1}{m_\pi}\right)^3,$$

we arrive at $T_0 \approx 185$ MeV and $a \approx 4 \cdot 10^3$ MeV$^{\frac{3}{2}}$.

We shall see in the next Section that, indeed, a, α and T_0 will have values very near to these.

That our theory shows a behaviour so similar to that of a model in which all particles are distinguishable (that they have zero mass is irrelevant) is not surprising, because with the exponentially increasing mass spectrum there are for $E \to \infty$ so many particle states available that it almost never happens that two equal ones are present at the same time. Thus, all those which are actually present can be distinguished from each other.

4. – Physical interpretation.

4·1. *The highest temperature* T_0. – Our self-consistency requirement has forced us rather unexpectedly into an exponentially growing mass spectrum and a highest temperature T_0. The question is now: does nature indeed behave like this? If so, can we find reasonable values for the parameters which we could not yet determine from conditions inside our theory? It will be shown that both are very likely to be true.

The first important result is the highest temperature. Are there experimental indications of its existence? Assume that there is indeed such a temperature T_0. Then in all collisions of hadrons with sufficiently large total energy and momentum transfer, the temperature T_0 will be reached but, for $E \to \infty$, never overpassed. Without going into any details of a theory which still has to be developed, one would expect that this temperature must govern the transverse momentum distribution of the outgoing particles; that it must be the *transverse* momentum distribution is clear, because this distribution will not be affected by any kinematical effect caused by fast, and from collision to collision enormously varying, relative motions of different parts of the heated volume (*): any Lorentz transformation in the direction of the collision axis will leave $p_\perp$ and its distribution invariant.

Because of this invariance we may calculate for $T \approx T_0$ the transverse momentum distribution for the simple case of an ideal equilibrium in a volume at rest (in that case the amount of energy transformed into heat would be maximal, but the temperature would be T_0 and that is all what counts for the momentum distribution). From the usual formulae of statistical mechanics and with $x_{\alpha k} \equiv \exp[-\varepsilon_{\alpha k}/T]$ our partition function

$$Z = \sum_{(\nu)} \prod_{\alpha k} x_{\alpha k}^{\nu_{\alpha k}}, \qquad k = \begin{cases} \beta = 0, 1, 2, \dots, \infty & \text{for bosons,} \\ \varphi = 0, 1 & \text{for fermions,} \end{cases}$$

(here the ground state $\nu_{\alpha k} \equiv 0$ is not subtracted) leads to the average occupation numbers

$$\bar{\nu}_{\alpha k} = x_{\alpha k} \frac{\partial}{\partial x_{\alpha k}} \log Z = \begin{cases} \dfrac{1}{\exp\left[(1/T)\sqrt{p_\alpha^2 + m_k^2}\right] - 1} & \text{for bosons,} \\[2ex] \dfrac{1}{\exp\left[(1/T)\sqrt{p_\alpha^2 + m_k^2}\right] + 1} & \text{for fermions.} \end{cases}$$

If we are interested in a definite (stable) mass, then by integrating over the δ-function corresponding to this mass in the mass spectrum and multiplying by the density

$$\frac{V_0}{2\pi^2}\, \mathrm{d}p_x\, \mathrm{d}p_y\, \mathrm{d}p_z$$

of states, we immediately obtain the momentum distribution; this may then

(*) As actually it will turn out that $T_0 \approx m_\pi$ is rather «low», T_0 will rapidly be approached in collisions long before thermodynamical equilibrium in the usual sense is reached. We thus may have an almost constant temperature T_0 all over the volume V_0 without necessarily having a constant energy density and without having transformed all kinetic energy in V_0 into heat motion. Remaining collective motions of whole parts of V_0 will then be strongly correlated to the former motion of the incoming colliding particles. In the longitudinal directions this collective motion will in general suppress the isotropic but small heat motion (except in very central collisions).

be integrated over the p_x component in order to obtain the transverse momentum distribution

$$(40) \qquad w_{\substack{\text{Bose}\\\text{Fermi}}}(p_\perp)\, \mathrm{d}p_\perp = \text{const}\cdot p_\perp \mathrm{d}p_\perp \cdot \int_0^\infty \mathrm{d}p_x \frac{1}{\exp\left[(1/T)\sqrt{p_x^2 + \mu^2}\right] \mp 1}, \qquad \mu^2 \equiv p_\perp^2 + m^2 .$$

Now $T \leqslant T_0$: Let us anticipate the result (43), namely that $T_0 \approx m_\pi$, in order to see that for $\mu = \sqrt{p_\perp^2 + m^2} \gg m_\pi$ (a few times m_π suffices) the ± 1 in the denominator of the integrand is irrelevant. This is certainly true for nucleons and not bad for all other particles with $m < m_N$ except for pions; for pions it will be valid once $p_\perp$ is larger than about $(2 \div 3)$ pion masses. Let us assume this; then the integral becomes—for boson as well as fermions—for $T \to T_0$

$$(41) \qquad w(p_\perp) \approx \text{const}\cdot p_\perp \cdot \int_0^\infty \mathrm{d}x \exp\left[-\frac{1}{T_0}\sqrt{x^2 + \mu^2}\right] = \text{const}\cdot p_\perp \cdot \sqrt{p_\perp^2 + m^2}\cdot K_1\left(\frac{\sqrt{p_\perp^2 + m^2}}{T_0}\right).$$

With the asymptotic formula for K_1 [the same as (14)], which may be used on account of our previous assumptions, we obtain

$$(42) \qquad w(p_\perp) \approx \text{const}\cdot p_\perp \sqrt{T_0 \sqrt{p_\perp^2 + m^2}} \exp\left[-\frac{1}{T_0}\sqrt{p_\perp^2 + m^2}\right] \to c \cdot p_\perp^{\frac{3}{2}} \exp\left[-\frac{p_\perp}{T_0}\right],$$

(the latter for $p_\perp \gg T_0$ *and* $\gg m$). In words: except for pions, where this holds only for $p_\perp$ larger than a few times m_π, the transversal momentum distribution will of the Boltzmann type (42).

This formula must apply even to elastic scattering (except in the diffraction region) for the following reasons:

1) treating elastic scattering by our formalism does *not* mean that we apply statistical mechanics to a two-body system (which would indeed be nonsense); it means in fact the following: our thermodynamical system is able to choose not only the momenta, but also the number and kinds of particles according to its convenience, that is: according to statistical distribution laws implied by the partition function. One possible choice is: two final particles.

2) $w(p_\perp)\,\mathrm{d}p_\perp$ as given by (42) is the probability that a particle chosen at random will have a transversal momentum between $p_\perp$ and $p_\perp + \mathrm{d}p_\perp$, no matter how many other particles there are and what they do.

3) we have seen that for sufficiently large energy and momentum transfer the temperature becomes sharp and tends to T_0^-. Therefore, a given fixed temperature (implying a Boltzmann distribution of $p_\perp$) and rigorous energy-momentum conservation (being most stringent in elastic scattering) are compatible.

4) the differential elastic cross-section (as a function of $p_\perp$) will then contain three types of factors:

i) $w(p_\perp)$ from (42),

ii) the probability that the number of particles is $N=2$,

iii) kinematical and geometrical factors (algebraic in $p_\perp$, E).

According to preliminary calculations $N(E)$ obeys a Poisson distribution with $\bar{N}\sim\log E$, so that $W(N=2)\sim(1/E)(\log E)^2$ is a factor which varies extremely slowly as compared to the exponential in (42), where $p_\perp\sim E$ for elastic scattering. The same holds for the comparison of the factors of type iii) to $w(p_\perp)$. The small elastic cross-sections outside the diffraction region therefore do not come from a small probability for two-body final states (the two final particles may be anything between the initial particles and two heavy fire-balls) but from the Boltzmann factor (42). All two-body final states would have such small weights, but there are very many of them [*e.g.*, $p+p\to p+p$, $p+\mathcal{N}^*$, $\mathcal{N}^*+\mathcal{N}^*$, $\pi+d$, etc.; all these should follow the Boltzmann law].

5) the elastic scattering is therefore one particular process of the many competing ones; its probability is calculated from the partition function and is mainly given by (42). For 90° scattering angle (42) becomes equivalent to eq. (1). Thus the large-angle elastic cross-section, as described by the usual statistical theory, is only one very particular case of the class of states described by our thermodynamical theory.

Our conclusion is then that in all high-energy events with sufficient total energy and momentum transfer—these events ranging from elastic scattering to jets with hundreds of secondaries—the transversal momentum distribution should be of the Boltzmann type (42) *with one and the same temperature* T_0 *which is independent of the primary energy, of the colliding particles (hadrons), and of the multiplicity* (*). [In jets a slow apparent increase of T_0 with the primary energy could be tolerated: it would be the effect of smearing out the spectrum by successive decays of the emitted fire-balls into smaller and smaller ones until finally the observed pions remain, having a transversal spectrum with an effective temperature $T_0'\geqslant T_0$].

(*) In collisions of unlike particles (*e.g.* $\pi+p$) our formalism does not allow for a forward-backward asymmetry, which has to do with the collective motion (footnote p. 163) surviving the struggle against thermodynamic equilibrium. To explain this and to choose a more suitable variable than $p_\perp$ is one of the remaining nontrivial problems.

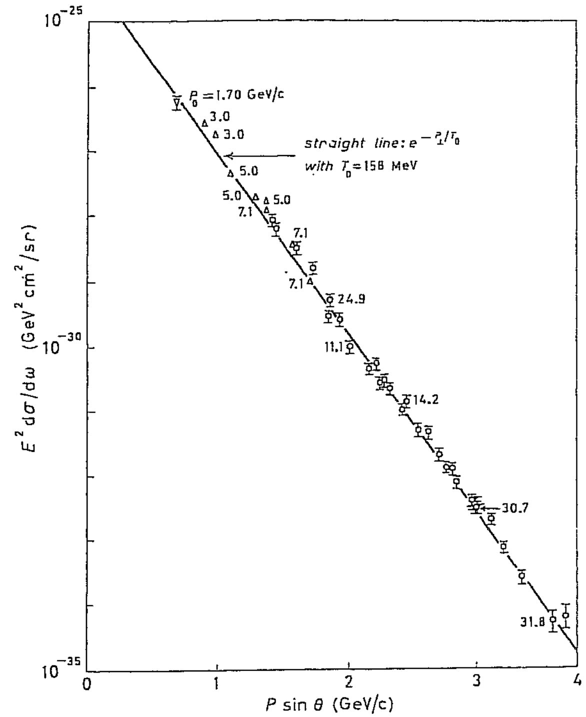

Fig. 1.

This conclusion seems to be in good agreement with the observed facts. A review with further references was recently presented by OREAR [14]. The elastic scattering which shows no apparent broadening due to decay is best suited for determining T_0. There, eq. (42) fits indeed very well the experiment over a region where $w(p_\perp)$ varies by nine orders of magnitude! [Fig. 1]. The electromagnetic form factors of the proton seem to behave accordingly, this was pointed out by WU and YANG [15]. All this, and up to jets with 10^6 GeV

([14]) J. OREAR: *Phys. Lett.*, 13, 190 (1964) and private communication.

([15]) T. T. WU and C. N. YANG: *Some speculations concerning high-energy large momentum transfer processes*, Brookhaven National Laboratory, preprint, September 1964. (submitted to *Phys. Rev.*).

primary energy, with T_0 values which vary only little from experiment to experiment and always lie around 160 MeV. Also our numerical phase-space calculations yielded such a value, namely 150 MeV [compare eq. (1)]. We shall take the most reliable experimental value coming from elastic scattering [14]

(43) $$T_0 = 158 \pm 3\,[\mathrm{MeV}]\,.$$

On the basis of the experimental evidence presented by OREAR and by WU and YANG we believe that the question: «is there a highest temperature T_0 in nature?» can be answered by «yes» and that the value of T_0 is rather well fixed by eq. (43).

4·2. *The other parameters. The mass spectrum.* – Having fixed T_0, we remain with three other parameters of which b, the constant factor in $\sigma(E)$, is rather uninteresting. The important ones are any two of a, V_0 and α [related by (32)]. We can estimate a priori the approximate values of α and of V_0. Let us start with V_0. We have determined T_0 from processes in which actually the interacting system has been enclosed in a volume of the order of a nucleon volume and this is, indeed, the only volume which we ever could expect to be relevant in high-energy collisions, since the interaction ceases to exist for larger distances. Thus

$$V_0 = \frac{4\pi}{3}\left(\frac{1}{\mu}\right)^3,$$

where μ should not be very different from m_π. As T_0 itself is also of the order of m_π, we see that we actually come very near to what distinguishable particles of $m = 0$ would give: (37).

Next consider α. The density of states is according to (30)

(44) $$\sigma(E) \to bE^{\alpha-1}\exp[E/T_0]\,,$$

whereas $\varrho(m)$ is given by (26)

(45) $$\varrho(m) \to \frac{a}{m^{\frac{5}{2}}}\exp[m/T_0]\,.$$

Hence, for E, $m \to \infty$

(46) $$\frac{\varrho(x)}{\sigma(x)} \to \frac{a}{b}\cdot x^{1-\alpha-\frac{5}{2}}\,.$$

Recalling the discussion following the self-consistency condition (13) we would expect that $1-\alpha-\frac{5}{2} \approx -2$ or $\alpha \approx \frac{1}{2}$; if somehow it would turn out that α is between $\frac{1}{4}$ and 1 we would still be satisfied.

We shall now proceed as follows: we take the experimental mass spectrum and see whether it can be fitted by $\varrho(m) \to am^{-\frac{5}{2}}\exp[m/T_0]$ with $T_0 = 158$ MeV; and if so, with which value of a. Having determined this value, we see whether the relation (32)

(47) $$aT_0^{\frac{3}{2}} = (2\pi)^{\frac{3}{2}}\frac{\alpha}{V_0} = \sqrt{\frac{9\pi}{2}}\,\alpha\mu^3 = 3.75\alpha\mu^3$$

can be satisfied with values of α and V_0 corresponding to what we found a priori reasonable: $\alpha \approx \frac{1}{2}$; $\mu \approx m_\pi$.

That our *asymptotic* $\varrho(m)$ should, in fact, fit the mass spectrum in the region up to ≈ 1200 MeV (where it seems to be pretty well known experimentally) is not obvious and must be considered a lucky accident if it happens. It is, on the other hand, not unreasonable to expect this, because with T_0 being as it is (158 MeV), the exponential factor $\exp[m/T_0]$ should pratically govern the mass spectrum once m is several times larger than T_0, *i.e.*, already at ≈ 1000 MeV (unless the complete unknown factor $f(m)$ in front of the exponential would be very queer).

We have taken the mass spectrum as published by the Berkeley group [10] and smoothed it out, in order to obtain a function which can be compared to our asymptotic $\varrho(m)$. Without smoothing

(48) $$\varrho_{\exp}(m) = \sum_{i=1}^{N}\nu_i\delta(m-m_i)$$

would be the spectrum for stable particles and sharp resonances. The sum goes over Bose and Fermi particles; as we have seen, only the sum of $\varrho_B + \varrho_F$ enters into the partition function when $T \to T_0^-$, hence our $\varrho(m)$ in (45) is, in fact, $\varrho_B + \varrho_F$. Furthermore, our $\varrho(m)$ is in so far unconventional as it counts every state [see (4)]; consequently, each mass occurring in (48) has to be counted as often as it has states

(49) $$\nu_i = (2J_i+1)(2I_i+1)\cdot 2^{\lambda_i} \text{ with } \lambda_i = \begin{cases} 1 \text{ if particle} \neq \text{antiparticle},\\ 0 \text{ if particle} = \text{antiparticle}.\end{cases}$$

Here J and I are spin and isospin, respectively, of the particle. In order to obtain a smooth function, we have replaced in the above sum (48) the δ-functions by normalized Gauss functions. In Fig. 2 we plot for $m = 0, 200, 400, \ldots$, 2000 MeV the following function

(50) $$\bar{\varrho}_{\exp}(m) = \frac{1}{\sqrt{2\pi\tau^2}}\sum_{i=1}^{N}\nu_i\exp\left[-\frac{(m-m_i)^2}{2\tau^2}\right], \qquad \tau = 200\ \mathrm{MeV},$$

where the sum goes from the pion mass to the highest known resonances. Above

1200 MeV there is every reason to believe that our experimental knowledge is still very incomplete—in particular regarding bosons. Up to 1200 MeV the function seems to increase exponentially—with fluctuations, of course. The slope is well fitted by $T_0 = 158$ MeV. Also we have drawn our asymptotic $\varrho(m)$ with the value

(51) $$a = 6.45 \cdot 10^3 \text{ MeV}^{\frac{3}{2}} \text{ (see Fig. 2)}.$$

In comparing our curve with the experimental values one should keep in mind that our $\varrho(m)$ is an asymptotic formula for $m \to \infty$ which becomes wrong when the factor $m^{-\frac{5}{2}}$ begins to govern the behaviour (dashed line). Thus, a comparison is possible only in a narrow range around 1000 MeV—just enough to guess the value of a. As the figure shows, our $\varrho(m)$ seems to have a good chance to be the correct extrapolation to higher mass values (*). If so, then one should expect a great number of new resonances to be discovered above 1200 MeV. Somewhere, not far above 2000 MeV the resonances will probably become so dense that an experimental resolution seems hopeless and a true continuum starts. Here we should remark that there seems to be no obvious way to disentangle the number of resonances between m, $m + \mathrm{d}m$ from the number of states counted by our $\varrho(m)$; namely, our theory does not say how the density of resonances and the multiplicity per resonance $\nu = (2J+1)(2I+1) \cdot 2^{\lambda}$ [see (49)] separately increase with m (our $\varrho(m)$ counts the product of these two). It may be that on the first factor, $(2J+1)$, the Regge poles, and on the second $(2I+1)$, the higher symmetries will have to say something. In view of what we called a fire-ball [discussion below eq. (13a)] we should, however, expect that $J < 10$ (or even lower), whatever m is.

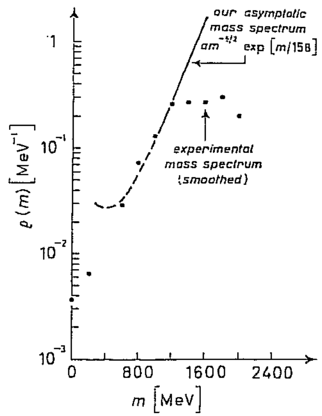

Fig. 2.

We now insert our values for a and T_0 into (47) and obtain

$$aT_0^{\frac{3}{2}} = 12.8 \cdot 10^6 \text{ (MeV}^3\text{)},$$

so that

$$\alpha\mu^3 = 3.42 \cdot 10^6 \text{ (MeV}^3\text{)}.$$

(*) It would be easy to find a $\varrho(m) = f(m) \cdot \exp[m/158]$ which practically coincides with our asymptotic form above 1000 MeV and follows the experimental values very well down to $m = 0$; such a $\varrho(m)$ would look very impressive in Fig. 2, but is physically not significant.

With $\alpha = \frac{1}{2}$ we obtain $\mu = 190 \text{ MeV} = 1.37\, m_\pi$ and with $\alpha = 1$ we find $\mu = 151 \text{ MeV} = 1.1\, m_\pi$. As we have anticipated on page 162 all parameters T_0, α, V_0 have turned out to be very near those for the model of massless distinguishable particles (accidentally?). In any case, the values are reasonable and all lie in the rather narrow range which is given by the condition (47) and the a priori argument that $\mu \approx m_\pi$ and $\alpha \approx \frac{1}{2}$. With this a priori limitation of two parameters the value of a was almost fixed and we have, indeed, only T_0 at our free disposal. With the one value $T_0 = 158$ MeV we could fit the transversal momentum distribution and a rather convincing extrapolation of the mass spectrum.

5. – Conclusions; open questions; speculations.

We have developed a thermodynamical theory of strong interactions which could be based on three postulates:

i) strong interactions are so strong that they produce an infinity of resonances (which for $m \to \infty$ are called fire-balls);

ii) fire-balls can be described by statistical thermodynamics;

iii) fire-balls consist of fire-balls.

These three postulates fix completely the structure of the theory; some numerical values of parameters remain, however. undermined. Except for one of them, the highest temperature T_0, a priori limitations have been given which fix them almost completely. The simple model of a gas of distinguishable particles, which shares many features with our present theory (but should not be taken seriously), leads to a determination of T_0 which agrees rather well with the experimental value. This seems to indicate that by only slightly narrowing the above postulates one might be able to calculate T_0 also in the framework of the present theory. It could be that this narrowing consists in replacing the first of the above postulates by another one, which may be formulated (tentatively):

i′) strong interactions are as strong as they possibly can be without violating postulate ii).

Similar conjectures have been pronunced already by several authors and in several versions [16] without reference to thermodynamics, however. In our

([16]) See *e.g.*, G. F. Chew and S. C. Frautschi: *Phys. Rev. Lett.*, 5, 580 (1960); *Phys. Rev.*, 123, 1478 (1961). More implicitly the «maximum strength» is contained in results by M. Froissart, A. Martin, N. N. Meiman, which should be seen together: A. Martin (private communication) showed in a manner similar to that used by N. N.

case the point is this:

we have met here an example of extreme behaviour, namely: the mass spectrum grows exponentially. As we have seen this is the maximum tolerated—otherwise postulate ii) would have been violated. This maximal growth was enforced by postulate iii); but this postulate has been guessed from experience. There was no a priori reason to expect it to hold (and we know other interactions where it does not). After our analysis it is clear that this postulate—or the experimental behaviour on which it was based—is equivalent to the extremal qualities of the mass spectrum. The rate of increase of the mass spectrum has certainly to do with the strength of the underlying interaction (and with its structure). It is conceivable that with greater strength of the interaction the mass spectrum could have grown faster than exponentially and thermodynamics would have broken down. This alone is probably not a sufficient reason to convince nature that it should renounce such a strong interaction. It is interesting, however, to observe that thermodynamics, here on the borderline of its domain of existence, implies another extremal behaviour: that of the elastic-scattering amplitude. As we have seen, the transversal momentum distribution even in elastic scattering is asymptotically a Boltzmann distribution with T_0 as temperature. Thus, we conclude for the elastic-scattering amplitude that in the physical region (s-channel; scattering of two equal particles)

$$\log A(s, t=-as) \xrightarrow[s\to\infty]{} -\frac{\sqrt{s}}{2T_0}\sqrt{a(1-a)}\,, \qquad 0<a<1.$$

It has been shown [17] that no faster than exponential decrease is permitted (supposing reasonable analytical properties of $A(s, t)$) unless the amplitude becomes identically zero. This behaviour also is the limiting one which just allows to solve uniquely partial-wave dispersion relations if the discontinuity along the left-hand cut is given [18].

Meimann (see below) that the squared coupling constant is smaller than some integral over forward elastic $\pi\mathcal{N}$ cross-sections; which is (almost) a statement about f^2 being smaller than some integral over the total cross-sections: increasing f^2 means increasing σ_{tot}. Now, M. Froissart and later A. Martin (with weaker assumptions) proved $\sigma_{tot}(s\to\infty)\leqslant \text{const}\,(\log s)^2$; experimentally, it seems to stay constant which is only little less than the maximum possible increase. For details see: A. Martin: *An absolute bound on the pion-pion scattering amplitude*, ITP-134 Stanford University, July 1964 (unpublished); M. Froissart: *Phys. Rev.*, **123**, 1053 (1961); A. Martin: *Phys. Rev.*, **129**, 1432 (1963); N. N. Meimann: *Žurn. Ėksp. Teor. Fiz.*, **44**, 1228 (1963) (translated *Sov. Phys. JEPT*, **17**, 830 (1963).

(17) A. Martin: *Minimal interactions at very high transfer*, and further references given there; CERN preprint 9887/TH. 490, 30 November 1964.

(18) A. P. Balachandran: *Consequences of the strong asymptotic decrese of the fixed-angle scattering amplitude*, University of Chicago, preprint EFINS 64-40, July 1964.

I cannot believe that all this is a purely accidental coincidence; meeting with extremal properties in so different respects seems to indicate that strong interactions are indeed governed by some extremal principle which is not yet understood. Accepting this point of view, it seems possible to guess also the numerical value of T_0 without recurrence to the experiment. The argument would be: among several *types* of functions the exponential growth of the mass spectrum is maximal. Once the exponential type is chosen, namely

$$\log \varrho(m) \to m/m_0\,,$$

we again take the *maximal* increase which we could obtain by relating the free parameter m_0 to any of the fundamental mass values occurring in strong interactions. With this argument we find $m_0 = m_\pi = T_0$. That the experimental value is somewhat (by a factor 1.14) larger could be understood as the influence of the less strong interactions related to strange particles, whose smallest mass is m_K.

We do not pretend that the above remarks give an explanation to any of the more fundamental questions about strong interactions; we believe, however, that thermodynamics adds another indication and offers a new view. On the other hand, one has to face the possibility that not much more than thermodynamics can be done at high energies and that even the analytical properties of scattering amplitudes are determined by thermodynamics, unitarity and crossing relations.

If our present theory is correct, then it might be of not much use to go in *individual* scattering experiments to much higher energies than a few GeV, because above that only the number and longitudinal momentum of the secondaries produced would increase, whereas the details of interaction would be hidden under the Boltzmann distribution; even the mass distribution of secondaries follows a probability distribution and would not reveal any of the secrets we are after—except for one possibility: the basic triplets making up the nucleons. Their possible existence has presumably no implications on the present theory; they would just add one line somewhere in our continuous mass spectrum.

Our belief that thermodynamics with a highest temperature $T_0 \approx m_\pi$ indeed governs strong interactions at sufficiently high energy and momentum transfer is supported also by the observation of Wu and Yang [15] that the electromagnetic form factors of the proton might and, in fact, seem to depend on $\sqrt{q^2}$, such that

$$\log F(q^2) \xrightarrow[q^2\to\infty]{} -\frac{\sqrt{q^2}}{4T_0}\,,$$

namely, as the 4th root of the elastic pp cross-section. Our present thermodynamics would, indeed, provide

«the mechanism independent of the method of excitation»

which WU and YANG postulate in order to conjecture the above behaviour of the form factors as well as of various cross-sections.

It is likely that *pure* electromagnetic and weak interactions do not show the thermodynamic behaviour described in this paper—at least not as long as no real hadrons are produced. It cannot, however, be exluded that even below the threshold for hadron production some thermodynamical features creep in via virtual hadrons. In that case T_0 would somehow appear in electron-electron scattering (and other such interactions) at sufficiently high energy and momentum transfer. Unfortunately, the centre-of-mass energy of such a collision should be in the GeV region (which in the next future seems out of question).

Our theory might also have consequences in astropysics, where the model of the «neutron star» (*) would perhaps be improved by including all possible hadrons—*i.e.*, by adoption of the present thermodynamics. A first step in this direction has already been made by AMBARTSUMIAN and SAAKYAN [20].

It remains to mention a few open problems *inside* our theory:

i) We have not explained why resonances and fire-balls take part in the «equilibrium» as if they were stable particles.

ii) We have only calculated the transversal momentum distribution. In doing so we assumed a rather peculiar thermodynamical «equilibrium» where still large collective relative motions in the direction of the collision axis remained and where the word equilibrium merely meant constant temperature $\approx T_0$ but not constant energy density. No attempt was made to describe this state of affairs in detail.

iii) Related to the foregoing point is the still lacking theory of jets. It will certainly not be a two-fire-ball theory. Some kinematical model will be unavoidable, because thermodynamics alone can never explain a forward-backward asymmetry in collisions of unlike particles; it may be that introducing the impact parameter will already suffice to calculate multiplicities and longitudinal momentum distribution. Such a theory would also provide a better justification for the application of thermodynamics to elastic scattering.

(*) For a review and further references see *e.g.* CHIU [19]. I am grateful to Prof. G. COCCONI for having drawn my attention to this possible application.

[19] H. Y. CHIU: *Ann. Phys.*, **26**, 364 (1964).

[20] V. A. AMBARTSUMIAN and G. S. SAAKYAN: *Sov. Astr. Journ.*, **5**, 601 (1962).

iv) The relation between thermodynamics and analytical properties of the scattering amplitude is still obscure. The bridge between these two is unitarity, where a sum over the mass spectrum is implied.

* * *

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APPENDIX I

Convergence of the sum in eq. (12).

Consider the integral

$$\int_0^\infty \varrho(m;\,n)\,m^2 K_2(nsm)\,\mathrm{d}m \equiv I(n,\,s)\,. \tag{A1.1}$$

For all $0<x<\infty$ we have [11]

$$\begin{cases} K_2(x)=\int_0^\infty \exp[-x\cosh t]\cosh 2t\,\mathrm{d}t<\int_0^\infty \exp[-x\cosh t]\cosh \frac{5}{2}t\,\mathrm{d}t=K_{2+\frac{1}{2}}(x)\,, \\ K_{2+\frac{1}{2}}(x)=\exp[-x]\sqrt{\frac{\pi}{2x}}\cdot\left(1+\frac{3}{x}+\frac{3}{x^2}\right) \end{cases} \tag{A1.2}$$

Using the fact that $\varrho(m)=0$ for $m<m_\pi$, we find with $\varrho(m;\,n)\leqslant\varrho(m)$ and using (A1.2)

$$I(n,\,s)<\int_0^\infty \varrho(m)\sqrt{\frac{\pi}{2nsm}}\exp[-nsm]\left(m^2+\frac{3m}{ns}+\frac{3}{n^2s^2}\right)\mathrm{d}m\leqslant$$

$$\leqslant\int_0^\infty \varrho(m)\sqrt{\frac{\pi}{2sm}}\exp[-sm]\left(m^2+\frac{3m}{s}+\frac{3}{s^2}\right)\mathrm{d}m\equiv F(s)\,.$$

Hence $I(n,\,s)<F(s)$ where $F(s)$ is independent of n. Therefore, in eq. (12)

$$\sum_{n=1}^\infty \frac{1}{n^2}I(n,\,s)<F(s)\sum_{n=1}^\infty\frac{1}{n^2}=F(s)\cdot\zeta(2)\,, \tag{A1.3}$$

that is, if the first integral $(n=1)$ converges, then the sum converges.

Assume now that the first integral ($n=1$) does not converge but that the others ($n\geqslant 2$) do. Since the exponential factor in the asymptotic form of $K_p(x)$ does not depend on p and is always $\exp[-x]$, it follows that as long as $\varrho(m)$ grows less than exponentially all integrals $I(n, s)$ are finite for $s>0$ and diverge for $s\to 0^+$. Hence at $s=0$ it is not possible to have the first integral $=\infty$ and the others finite. This is possible, however, at $s=s_0>0$. In order to have a divergent integral for $s\to s_0^+$, $\varrho(m)$ must grow exponentially

$$\text{(A1.4)}\qquad \varrho(m)=f(m)\exp[s_0 m], \qquad f(m)=o(\exp[am]) \text{ for every } a>0 .$$

In that case

$$\text{(A1.5)}\qquad I(n, s)<\int_0^\infty \sqrt{\frac{\pi}{2nsm}}\, f(m)\left(m^2+\frac{3m}{ns}+\frac{3}{n^2s^2}\right)\exp[-m(ns-s_0)]\,\mathrm{d}m$$

Now we can have at $s=s_0$ a divergent integral for $n=1$ but convergent integrals for $n\geqslant 2$ (of course, $f(m)$ can be such that even for $n=1$ the integral converges at $s=s_0$). Then for $n\geqslant 2$ and $s\geqslant s_0$

$$I(n, s)<\int_0^\infty \sqrt{\frac{\pi}{4s_0m}}\, f(m)\left(m^2+\frac{3m}{2s_0}+\frac{3}{4s_0^2}\right)\exp[-ms_0]\,\mathrm{d}m\equiv G(s_0) .$$

Therefore in (12) for $s>s_0$

$$\sum_{n=0}^{\infty}\frac{1}{n^2}I(n, s)<I(1, s)+G(s_0)[\zeta(2)-1] .$$

Now for $s\to s_0^+$ the first term $I(1, s)$ may diverge, but not the rest of the sum.

APPENDIX II

Notations used for comparing functions.

1) If $|f(x)|\leqslant K\cdot g(x)$ for $x\to x_0$ we write $f(x)=O(g(x))$ for $x\to x_0$;

   « $f(x)$ is at most of the order of $g(x)$ »

2) If $f(x)/g(x)\to A$ for $x\to x_0$ we write $f(x)\to Ag(x)$ for $x\to x_0$;

   « $f(x)$ tends asymptotically to $Ag(x)$ »

3) If $f(x)/g(x)\to 0$ for $x\to x_0$ we write $f(x)=o(g(x))$ for $x\to x_0$;

   « $f(x)$ is of smaller order than $g(x)$ ».

4) In the particular case that the function $f(x)$ is compared to an exponential function we say:

   a) « $f(x)$ grows faster than exponentially »,

      if $f(x)\exp[-ax]\to\infty$ for every $a>0$ when $x\to\infty$.

   b) « $f(x)$ grows exponentially »,

      if an $a_0>0$ exists such that

$$f(x)\exp[-ax]\begin{cases}\to 0 & \text{for } a>a_0\\ \to\infty & \text{for } a<a_0\end{cases}\qquad \text{when } x\to\infty .$$

      It remains to be stated, in individual cases, what happens for $a=a_0$.

   c) « $f(x)$ grows less than exponentially »,

      if $f(x)\exp[-ax]\to 0$ for every $a>0$ when $x\to\infty$.

5) If we wish to distinguish between x approaching x_0 from below and from above, we write

   $x\to x_0^-$ if x approaches x_0 from below,

   $x\to x_0^+$ if x approaches x_0 from above.

APPENDIX III

Nonexistence of nonexponential solutions.

We show here that our eqs. (12), (13) do not possess a solution $\{\sigma(x), \varrho(x)\}$ which grows less than exponentially for $x\to\infty$.

We first simplify the notation a little by writing for the exponent in eq. (12)

$$\text{(A3.1)}\qquad \frac{V_0}{2\pi^2 s}\sum_1^\infty\frac{1}{n^2}\int_0^\infty m^2\varrho(m; n)K_2(nsm)\,\mathrm{d}m\equiv\frac{V_0}{(2\pi)^{\frac{3}{2}}}\cdot\frac{1}{s^3}\int_0^\infty m^{\frac{3}{2}}\varrho(m)Q(sm)\,\mathrm{d}m$$

by which we have introduced the function

$$\text{(A3.2)}\qquad \begin{cases} Q(sm)\equiv\sqrt{\dfrac{2}{\pi}}\,s^2\sqrt{m}\displaystyle\sum_1^\infty\frac{1}{n^2}r(m; n)K_2(nms), & r(m, n)\equiv\begin{cases}1 & n \text{ odd},\\ \Delta\varrho/\varrho & n \text{ even},\end{cases}\\[2ex] \varrho(m)=\varrho_B(m)+\varrho_F(m), & \Delta\varrho(m)=\varrho_B(m)-\varrho_F(m) \text{ [see (}^6\text{)]} .\end{cases}$$

We shall need the asymptotic behaviour of $Q(sm)$ for large and small arguments.

From (14) we find

$$Q(sm)\begin{cases}\nearrow \sqrt{\dfrac{2}{\pi}}\,\dfrac{2}{m^{\frac{3}{2}}}\sum\limits_{1}^{\infty}\dfrac{r(m,n)}{n^{4}} & \text{for } sm\to 0,\\[2ex] \searrow s^{\frac{3}{2}}\exp[-sm] & \text{for } sm\to\infty.\end{cases}\tag{A3.3}$$

With these notations eq. (12) becomes

$$\begin{cases} Z(s)=\exp\left[\dfrac{V_0}{(2\pi)^{\frac{3}{2}}}\cdot\dfrac{1}{s^3}\cdot\displaystyle\int_0^\infty m^{\frac{3}{2}}\varrho(m)\,Q(sm)\,\mathrm{d}m\right]-1\,,\\[3ex] Z(s)=\displaystyle\int_0^\infty\sigma(E)\exp[-sE]\,\mathrm{d}E\,.\end{cases}\tag{A3.4}$$

The self-consistency condition requires that we find a pair of positive functions fulfilling (A3.4) and having the property $\log\sigma(x)\to\log\varrho(x)$ for $x\to\infty$. This can also be written

$$\varrho(x)=\sigma(x)\cdot f(x)\qquad\text{where}\qquad \log f(x)=o(\log\sigma(x))\,.\tag{A3.5}$$

As we wish to prove that this is impossible unless ϱ and σ grow exponentially, we shall restrict ourselves to functions which grow less than exponentially and show that the assumption, that among them a pair obeying (A3.4) and (A3.5) could exist, will lead to a contradiction. We, of course, exclude the trivial solution $\varrho=\sigma=0$.

Let us start with a $\varrho(m)$ which describes just the pions, *i.e.*, $\varrho(m)=3\delta(m-m_\pi)$. We obtain from (A3.4)

$$Z(s)=\exp\left[\frac{V_0}{(2\pi)^{\frac{3}{2}}}\cdot\frac{3}{s^3}\cdot m_\pi^{\frac{3}{2}}Q(sm_\pi)\right]-1\,,$$

and with $s\to 0$ the asymptotic behaviour

$$Z(s)\to\exp\left[\frac{3V_0\zeta(4)}{\pi^2 s^3}\right]\qquad\text{(for pions only)},\tag{A3.6}$$

where (A3.3) has been used together with $r(m_\pi,n)=1$ for all n. The result is the well-known partition function for a gas of mass-zero bosons with 3 degrees of freedom. We invert this $Z(s)$ and calculate the corresponding $\sigma(E)$:

$$\sigma(E)=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}Z(x+iy)\exp[(x+iy)E]\,\mathrm{d}(iy)\,,\qquad x>0.\tag{A3.7}$$

This integral is independent of x and indeed represents $\sigma(E)$, since we know

that $Z(s)$ can be written as a Laplace transform and the conditions for the validity of (A3.7) are fulfilled [Doetsch, Satz 3, I, 4.4]. As the integrand is holomorphic in the half-plane $x>0$ and—with our present $Z(s)$—has one single minimum on the real axis at some point $s_0(E)$, we may take the path of integration $-i\infty$ to $+i\infty$ through this $s_0(E)$ where it will have a maximum (saddle point method). We shall write

$$Z(s)\equiv\exp[h(s)]\,,\tag{A3.8}$$

and obtain asymptotically for $E\to\infty$ [replacing the integrand by a suitable Gauss function—Doetsch II, Chapter 3, § 5]

$$\begin{cases}\sigma(E)\xrightarrow[E\to\infty]{} Z(s_0(E))\exp[Es_0(E)]\cdot\sqrt{\dfrac{1}{2\pi h''(s_0)}}\,,\\[2ex] s_0(E)\text{ given by } h'(s_0)=-E.\end{cases}\tag{A3.9}$$

Equation (A3.6) then gives

$$\begin{cases}\sigma(E)\xrightarrow[E\to\infty]{}\sqrt{\dfrac{(3c)^{1/4}}{8\pi\cdot E^{5/4}}}\cdot\exp\left[\dfrac{4}{3}(3c)^{1/4}\cdot E^{3/4}\right],\\[2ex] c\equiv\dfrac{3V_0\zeta(4)}{\pi^2}\end{cases}\tag{A3.10}$$

As it should, the exponent equals the entropy for a massless Bose gas with 3 degrees of freedom.

We now observe that $\sigma(E)$ grows almost exponentially, namely, $\sim\exp[E^{\frac{3}{4}}]$, although we started from a $\varrho(m)$ which was just a δ-function. Clearly, any $\varrho(m)$ which goes to a constant for $m\to 0$ and for which $\int_0^\infty\varrho(m)\,dm$ exists, will lead to

$$\int_0^\infty m^{\frac{3}{2}}\varrho(m)Q(sm)\,\mathrm{d}m=f(s)\to\text{const}\qquad\text{for } s\to 0 \text{ [see (A3.3)]},$$

and will give the same $\sigma(E)$ as above, except for another value of c in eq. (A3.10). In other words: no such $\varrho(x)$ can be a solution. Indeed, since $\varrho(x)$ should be somewhere $\neq 0$, it follows at once that it must grow at least as $\exp[m^{\frac{3}{4}}]$ in order to have a chance to be a solution. We therefore restrict our further considerations to functions with the asymptotic behaviour

$$\exp[x^{\frac{1}{2}}]=o(\varrho(x))=o(\exp[ax])\qquad\text{for every } a>0.\tag{A3.11}$$

We have chosen $x^{\frac{1}{2}}$ in the l.h.s. because $x^{\frac{3}{4}}$ would have excluded, *e.g.*, $m^{-2}e^{m^{\frac{3}{4}}}$.

We must now evaluate the integral in the exponent of (A3.4) for such $\varrho(m)$

$$\int_0^\infty m^{\frac{3}{2}}\varrho(m)Q(sm)\,\mathrm{d}m = F(s)\,. \tag{A3.12}$$

So far, when $\int \varrho(m)\,\mathrm{d}m < \infty$, this $F(s) \to \text{const}$ for $s \to 0$. Indeed, $F(s) \approx \int \varrho(m)\,\mathrm{d}m$, because in the integrand all contributions coming from $m > M$ (M some suitable constant) are negligible and therefore, when $s \to 0$, the upper line of eq. (A3.3) becomes valid for the whole integration. Now, with functions of the type described by (A3.11), it is the lower line of (A3.3) which will determine the behaviour of the integral, because the main contributions to the integral come from a region around $m \approx m_0$, where the integrand has a maximum; for the functions considered, not only $m_0 \to \infty$, but even $sm_0 \to \infty$ when $s \to 0^+$. Hence, the asymptotic form of $Q(ms)$ for large argument is valid.

We choose—for s fixed—a very large mass M and write

$$\int_0^\infty m^{\frac{3}{2}}\varrho(m)\,Q(sm)\,\mathrm{d}m \equiv \int_0^M m^{\frac{3}{2}}\varrho(m)Q(sm)\,\mathrm{d}m + \int_M^\infty m^{\frac{3}{2}}\varrho(m)\,q(sm)\,\mathrm{d}m + \tag{A3.13}$$
$$+\, s^{\frac{3}{2}}\int_M^\infty m^{\frac{3}{2}}\varrho(m)\,\exp[-sm]\,\mathrm{d}m\,,$$

where

$$q(sm) \equiv Q(sm) - s^{\frac{3}{2}}\exp[-sm] = o\big(s^{\frac{3}{2}}\exp[-sm]\big) \tag{A3.14}$$

[in fact $q(sm) \sim (1/sm)s^{\frac{3}{2}}\exp[-sm]$; see (14)]. Now, the first integral, for $s \to 0^+$, becomes a constant; the second and third will diverge, but the second integral will have divergence of a lower order

$$\int(\text{second}) = o\left[\int(\text{third})\right]$$

and will, together with the first one, be neglected:

$$\int_0^\infty m^{\frac{3}{2}}\varrho(m)Q(sm)\,\mathrm{d}m \xrightarrow[s\to 0^+]{} s^{\frac{3}{2}}\int_0^\infty m^{\frac{3}{2}}\varrho(m)\,\exp[-sm]\,\mathrm{d}m \equiv s^{\frac{3}{2}}\bar{Z}(s)\,, \tag{A3.15}$$

where reintroducing the lower limit 0 has no effect on the asymptotic behaviour. Equation (A3.15) defines the abbreviation $\bar{Z}(s)$.

In this integral the integrand has a maximum at some place $m_0(s)$, which moves to ∞ when $s \to 0^+$, whereas the value and second derivative at the maximum value both diverge for $s \to 0^+$ (of course, if $\varrho(m)$ is of the type (A3.11)). In that case a formula similar to (A3.9) gives the asymptotic behaviour

of the integral; we write

$$m^{\frac{3}{2}}\varrho(m) \equiv \exp[g(m)] \tag{A3.16}$$

and obtain [replacing the integrand by a suitable Gauss function—Doetsch II, chapter 3, § 5]

$$\left\{\begin{array}{l} \bar{Z}(s) \xrightarrow[s\to 0^+]{} \exp[g(m_0(s)) - sm_0(s)]\cdot\sqrt{\dfrac{-2\pi}{g''(m_0)}}\,, \\ m_0(s) \text{ given by } g'(m_0) = s\,. \end{array}\right. \tag{A3.9'}$$

[It is reassuring that for the type of functions ϱ, σ considered here, a twofold application of formulae like (A3.9) and (A3.9′) leads from $Z(s)$ over $\sigma(E)$ back to $Z(s)$ in the asymptotic limit $s \to 0$.] Now comes the main conclusion. Suppose a solution of the type (A3.11) exists. Then $\varrho(x) = f(x)\sigma(x)$ with $\log f(x) = o(\log\sigma(x))$. We have defined (A3.16)

$$\left\{\begin{array}{l} x^{\frac{3}{2}}\varrho(x) = \exp[g(x)]\,, \\ \text{hence} \\ \sigma(x) \quad = \exp[g(x) - \frac{3}{2}\log x - \log f(x)]\,. \end{array}\right. \tag{A3.17}$$

For functions of the type (A3.11) we have

$$x^{\frac{1}{2}} = o(g(x)) = o(ax) \tag{A3.11'}$$

so that $\log x$ is also to be neglected asymptotically. We have by definition

$$Z(s) = \int_0^\infty \sigma(x)\,\exp[-sx]\,\mathrm{d}x\,.$$

With (A3.17) and the procedure corresponding to eq. (A3.9′) we obtain

$$\left\{\begin{array}{l} Z(s) \xrightarrow[s\to 0^+]{} \exp[g(x_0(s)) - \underbrace{\tfrac{3}{2}\log x_0 - \log f(x_0)}_{\text{to be neglected for } s\to 0^+} - sx_0]\cdot\sqrt{\dfrac{-2\pi}{g''(x_0)}}\,, \\ x_0(s) \text{ is given by } g'(x_0(s)) = s \text{ for } s \to 0^+. \end{array}\right. \tag{A3.18}$$

Now, to calculate the mass integral (A3.15) we use (A3.9′) and find

$$\left\{\begin{array}{l} \bar{Z}(s) \xrightarrow[s\to 0^+]{} \exp[g(x_0) - sx_0]\cdot\sqrt{\dfrac{-2\pi}{g''(x_0)}}\,, \\ x_0(s) \text{ as in (A3.18)}\,. \end{array}\right. \tag{A3.19}$$

According to (A3.15), the mass integral equals $s^{\frac{3}{2}}\bar{Z}(s)$. Inserting this into eq. (A3.4) (where the -1 can be neglected) we obtain

$$Z(s) \xrightarrow[s\to 0^+]{} \exp\left[\frac{V_0}{(2\pi)^{\frac{3}{2}}}\cdot\frac{1}{s^{\frac{3}{2}}}\cdot\bar{Z}(s) + \text{terms of smaller order}\right]. \tag{A3.20}$$

Taking two times the logarithm of (A3.20) we find

$$\log\log Z(s) \xrightarrow[s\to 0^+]{} \log\bar{Z}(s) + o(\log\bar{Z}(s)) \xrightarrow[s\to 0^+]{} \log\bar{Z}(s)\,.$$

From eqs. (A3.18) and (A3.19) we see that

$$\log\bar{Z}(s) \xrightarrow[s\to 0^+]{} \log Z(s)\,,$$

hence

$$\log\log Z(s) \to \log Z(s) \qquad \text{for } s\to 0^+, \tag{A3.21}$$

which is impossible. Thus no solution $\varrho(x) = o(\exp[ax])$ can exist.

Appendix IV

Existence of exponential solutions.

We start from eq. (23)

$$Z(s) = \exp\left[\frac{V_0}{(2\pi)^{\frac{3}{2}}}\cdot\frac{1}{s^3}\cdot\int_0^\infty m^{\frac{3}{2}}\varrho(m)Q(sm)\,\mathrm{d}m\right] - 1\,. \tag{A4.1}$$

Considering the asymptotic behaviour of $Q(sm)\to s^{\frac{3}{2}}\exp[-sm]$ and that of $\varrho(m)m^{\frac{3}{2}}\to f(m)\exp[s_0 m]$, we see that $Z(s)$ will, for $s-s_0\to 0^+$, take the asymptotic form

$$Z(z) \xrightarrow[z\to 0^+]{} \exp\left[\frac{V_0}{(2\pi)^{\frac{3}{2}}}\left(\frac{1}{s_0}\right)^{\frac{3}{2}}\cdot(Az^\alpha \text{ or } Bz^p\log z)\right] - 1\,, \tag{A4.2}$$

where $z\equiv s-s_0>0$; Az^α or $Bz^p\log z$ comes from the leading powers of $f(m)$ for $m\to\infty$. This is what theorem (22*a*-*c*) says. In fact, p takes only the values 1, 2, 3, ..., whereas these are excluded for α. On the other hand, writing $\sigma(E) = g(E)\exp[s_0 E]$ we obtain by the same theorem

$$Z(z) \xrightarrow[z\to 0^+]{} \text{const}\;[A' z^{\alpha'} \text{ or } B' z^{p'}\log z]\,. \tag{A4.3}$$

We may of course have several terms of that sort in both formulae, but for $z\to 0^+$ a definite one of them will be the leading term. We can at once discard the possibility that this is Az^α with $\alpha<0$, because then (A4.2) would diverge

as $\exp[c/z]^\beta$, $\beta>0$, and there is no such term in (A4.3). Thus certainly $\alpha>0$ but $\neq 1, 2, \ldots$.

It is, of course, the term $Bz^p\log z$ which attracts our attention—in particular when $p=0$, since then (A4.2) gives

$$\exp\left[\text{const}\cdot\log\frac{1}{z}\right] = \left(\frac{1}{z}\right)^{\text{const}},$$

which has a counterpart in (A4.3). We shall exploit this possibility soon. Let us, however, consider presently the case where $p\geqslant 1$ or where no such term is present at all. In that case z^α would be the leading power (*i.e.*, the less vanishing one):

$$Z(z) \xrightarrow[z\to 0^+]{} \exp[F(s_0) - Az^\alpha], \qquad \alpha>0. \tag{A4.4}$$

Clearly $A>0$, because for $z=0$ we have $F(s_0)$ in the exponent and when z grows, the exponent must decrease as it comes from an integral of the type $\int h(m)\exp[-zm]\,\mathrm{d}m$ with $h>0$. We obtain immediately a limitation on α by calculating $\bar{E}$ and the specific heat of our system by

$$\begin{cases} \bar{E} = -\dfrac{\mathrm{d}}{\mathrm{d}z}\log Z(z) = A\alpha z^{\alpha-1}, \\ -\dfrac{\mathrm{d}\bar{E}}{\mathrm{d}z}\,(\text{proportional to specific heat}) = -A\alpha(\alpha-1)z^{\alpha-2}. \end{cases} \tag{A4.5}$$

Of course, we do not accept a negative specific heat, hence $\alpha<1$ is necessary

$$0<\alpha<1 \qquad (\text{if } Az^\alpha \text{ is the «leading» term}) \tag{A4.6}$$

In that case $Z(z)$ is finite for $z\to 0^+$, but $\bar{E}(z)$ and the specific heat diverge for $z\to 0^+$. In so far the behaviour is quite similar to the one with the logarithmic term present (which shall be discussed later), it is, on the other hand, not so convincingly simple, because

$$Z(z) \xrightarrow[z\to 0^+]{} \text{const}\cdot\exp[-Az^\alpha] \equiv \int_0^\infty g(E)\exp[-zE]\,\mathrm{d}E \tag{A4.7}$$

would require a $g(E)$ which is not a polynomial. Of course, for $z\to 0^+$ we may expand

$$Z(z) \xrightarrow[z\to 0^+]{} \text{const}\cdot[1 - Az^\alpha + \tfrac{1}{2}A^2 z^{2\alpha}\ldots],$$

and with $\sigma(E)\to E^{-\alpha-1}\exp[s_0 E]$ we could produce the first and with similar further powers the following terms in the square bracket. But as z^α for $0<\alpha<1$ rises from 0 with infinite slope, the expansion in powers of z^α is not very good. This argument is not sufficient to reject the possibility of a solution where Az^α is the leading term in the exponential; but the one which we shall discuss in the following is so much more convincing that we presently discard Az^{α} as leading term.

We turn to the case that $m^{\frac{3}{2}}\varrho(m)$ is such that in (A4.2) terms $B_p z^p \log z$ appear and that in particular $B_0 \neq 0$. (In that case further terms of the form Az^{α}, $\alpha > 0$, are of course admitted but negligible for $z \to 0^+$.)

Going back to (A4.1) we see that what we obtain is not quite (A4.2) but, more accurately,

(A4.8)
$$Z(z) \xrightarrow[z\to 0^+]{} \exp\left[\text{const}\cdot\frac{1}{s^{\frac{3}{2}}}\cdot\log\frac{1}{z}\right] = \left(\frac{1}{z}\right)^{\text{const}/s^{\frac{3}{2}}},$$

since $s = s_0 + z$, this is not a pure power of z. In order to show that we can remove this difficulty and obtain a «pure» power of $1/z$ (not exactly, but with any desired precision) we must go back to the original form (12):

(A4.9)
$$Z(s) = \exp\left[\frac{V_0}{2\pi^2 s}\sum_1^{\infty}\frac{1}{n^2}\int_0^{\infty} m^2\varrho(m;\,n)\,K_2(nsm)\,\mathrm{d}m\right] - 1\,.$$

For $K_2(snm)$ we use an integral representation ([11]) which is valid in the whole snm plane cut from $-\infty$ to 0:

(A4.10)
$$K_2(nsm) = \sqrt{\frac{\pi}{2nsm}}\exp[-nsm]\cdot\int_0^{\infty}\exp[-t]\left(t+\frac{t^2}{2nsm}\right)^{\frac{3}{2}}\frac{\mathrm{d}t}{\Gamma(\frac{5}{2})}\,.$$

With $\varrho(m, n) = f(m, n)\exp[s_0 m]$ one has

$$\int m^2 f(m, n)\exp[s_0 m]\,K_2(nsm)\,\mathrm{d}m = \int_0^M + \int_M^{\infty} =$$

$$= \varphi(nsM) + \int_M^{\infty} f(m, n)\cdot\frac{4}{3}\,\frac{\exp[-(ns-s_0)m]}{\sqrt{2nsm}}\,\mathrm{d}m\int_0^{\infty}\exp[-t]\left(t+\frac{t^2}{2nsm}\right)^{\frac{3}{2}}\mathrm{d}t\,.$$

[illegible] $f(m, n)$ is such that for $s \to s_0^+$ the integral ($n=1$) does not converge (and [illegible] is supposed now) then also the other ($n = 2, 3, \ldots$) integrals have sin[illegible]ities, however, at the points $s_n = (s_0/n)$ which lie below s_0 and cannot [illegible]est us; thus, $\varphi(nsM)$ and all other integrals ($n \geqslant 2$) are holomorphic at [illegible] singularity $s = s_0$ of the $n=1$ integral. Therefore, $n > 1$ will be disregarded [illegible] now on.

[illegible] The leading terms in the exponent for $z \to 0^+$ will then come from the [illegible] integral, where $\varrho(m, 1) = \varrho(m)$

(A4.11)
$$\begin{cases} \dfrac{V_0}{2\pi^2 s}\displaystyle\int_0^{\infty} m^2\varrho(m)\,K_2(sm)\,\mathrm{d}m\,, \\[2ex] K_2(sm) = \sqrt{\dfrac{\pi}{2sm}}\exp[-sm]\cdot G(sm)\,, \end{cases}$$

where $G(sm)$ is the integral over t in (A4.10) with $n = 1$. It is holomorphic in the sm plane cut from $-\infty$ to 0 and has an asymptotic expansion (semi-convergent)

(A4.12)
$$G(sm) \xrightarrow[sm\to\infty]{} \sum_{k=0}^{N}\left(\frac{1}{2sm}\right)^k\frac{\Gamma(\frac{5}{2}+k)}{k!\,\Gamma(\frac{5}{2}-k)} + R_N(sm)\,.$$

With (A4.11) we find

(A4.13)
$$\frac{V_0}{2\pi^2 s}\int_0^{\infty} m^2\varrho(m)\,K_2(sm)\,\mathrm{d}m = \frac{V_0}{(2\pi)^{\frac{3}{2}}}\left(\frac{1}{s}\right)^{\frac{3}{2}}\int_0^{\infty} m^{\frac{3}{2}}\varrho(m)\exp[-sm]\,G(sm)\,\mathrm{d}m\,.$$

We again split the integral $\int_0^{\infty} = \int_0^M + \int_M^{\infty}$ and suppose M large enough so that with

(A4.14)
$$m^{\frac{3}{2}}\varrho(m) \equiv f(m)\exp[s_0 m]$$

the asymptotic form of $f(m)$

(A4.15)
$$f(m) \xrightarrow[m\to\infty]{} \sum_{n=1}\frac{f_n}{m^n}$$

can be used in the second integral. The first integral $\int_0^M$ contributes a function which is holomorphic in the s-plane cut from $-\infty$ to 0 and which does not interest us presently (it is essentially identical with $F(s, M)$, eq. (27)).

Denoting again $s - s_0 = z$, we obtain

(A4.16)
$$Z(z) \xrightarrow[z\to 0^+]{} \exp\left[F(z) + c\cdot\int_M^{\infty}\frac{G(sm)f(m)}{s^{\frac{3}{2}}}\exp[-zm]\,\mathrm{d}m\right].$$

In what follows we write

(A4.17)
$$\begin{cases} G(sm) \sim \displaystyle\sum_{k=0}\frac{g_k}{s^k}\cdot\frac{1}{m^k} & [g_k \text{ given by (A4.12)}]\,, \\[2ex] f(m) \sim \displaystyle\sum_{n=1}\frac{f_n}{m^n}\,, \end{cases}$$

where $\sim$ and the open upper limit of the sum indicate that these are (or may be) semi-convergent series. We obtain then

(A4.18)
$$\frac{G(sm)f(m)}{s^{\frac{3}{2}}} \sim \frac{1}{s^{\frac{3}{2}}}\sum_{r=1}\left[\sum_{n=1}^{r}\frac{g_{r-n}f_n}{s^{r-n}}\right]\frac{1}{m^r} \equiv \frac{1}{s^{\frac{3}{2}}}\sum_{r=1}\varphi_r(s)\,\frac{1}{m^r}\,,$$

which defines the abbreviation $\varphi_r(s)$. This gives after integration [theorem (22a-c)]

$$\text{(A4.19)} \quad \int_M^\infty \frac{G(sm)f(m)}{s^{\frac{3}{2}}} \exp[-zm]\,\mathrm{d}m \sim \left[\frac{1}{s^{\frac{3}{2}}} \sum_{r=1} \varphi_r(s) \frac{(-)^r}{(r-1)!} z^{r-1}\right] \log z \equiv h(z)\cdot \log z\,,$$

which defines the abbreviation $h(z)=[...]$. We wish to make $h(z)$ constant in order to obtain a pure logarithm in the exponent. Now with $s=z+s_0$ we have

$$h(z) = \left(\frac{1}{z+s_0}\right)^{\frac{3}{2}} \sum_{r=1} \varphi_r(z+s_0) \frac{(-)^r s_0^{r-1}}{(r-1)!} \left(\frac{z}{s_0}\right)^{r-1}$$

Inserting φ_r from (A4.18) and expanding the powers of $1/(z+s_0)$ yields

$$\text{(A4.20)} \qquad h(z) = \sum_{k=1} \left(\frac{z}{s_0}\right)^{k-1} \left[\sum_{r=1}^{k} \frac{(-)^r}{(r-1)!} \sum_{n=1}^{r} f_n g_{r-n} s_0^{n-\frac{5}{2}} \cdot \binom{-r+n-\frac{3}{2}}{k-r}\right].$$

We have to make the square bracket equal to a constant$\neq 0$ for $k=1$ and equal to 0 otherwise. As one sees, our freedom to choose the coefficients f_n is sufficient to achieve our aim up to any finite k (not up to $k=\infty$, as the series (A4.18) need not converge). Indeed, in each of the ensuing equations

$$\text{(A4.21)} \qquad \sum_{r=1}^{k} \frac{(-)^r}{(r-1)!} \sum_{n=1}^{r} f_n g_{r-n} s_0^{n-\frac{5}{2}} \binom{-r+n-\frac{3}{2}}{k-r} = \begin{cases} \text{const} & k=1, \\ 0 & k=2,3,\dots, \end{cases}$$

the label n reaches the value k as its highest value. Thus f_k occurs in the k-th equation but not in any earlier one. This f_k can be chosen to fulfil the equation. Hence,

$$\text{(A4.22)} \qquad \int_M^\infty \frac{G(sm)f(m)}{s^{\frac{3}{2}}} \exp[-zm]\,\mathrm{d}m = \text{const}\cdot\log\frac{1}{z} + h(z)\log\frac{1}{z},$$

where in $h(z)$ we can, by a proper choice of the coefficients f_n, push the exponent k_0 of its first nonvanishing power arbitrarily high up: $h(z)=az^{k_0}+\dots$. In other words, by a proper choice of the f_n we can achieve that

$$\text{(A4.23)} \qquad Z(z) \xrightarrow[z\to 0^+]{} \exp\left[F(z) + \text{const}\cdot\log\frac{1}{z}\right] \to \bar{F}(0)\cdot\left(\frac{1}{z}\right)^{\text{const}}$$

is valid for $z\to 0^+$ with any desired precision, although never exactly. We cannot, for instance, by choosing const=integer, make the cut in the z-plane, which goes from $z=0$ to $-\infty$, disappear; we can, however, make the discontinuity across this cut arbitrarily small in the neighbourhood of $z=0$.

But this is sufficient to guarantee that

$$Z(z) \xrightarrow[z\to 0^+]{} \bar{F}(0)\cdot\left(\frac{1}{z}\right)^{\text{const}}$$

can be written

$$Z(z) \to \int g(E)\exp[-zE]\,\mathrm{d}E\,,$$

where $g(E)$ behaves as a power for $E\to\infty$.

What we have done in the main text [eqs. (26)–(32)] is simply this: we have taken the first terms of $f(m)$ and of $G(sm)$

$$f(m) \sim \frac{a}{m} \quad (f_1 = a), \qquad \frac{G(sm)}{s^{\frac{3}{2}}} \sim \left(\frac{1}{s_0}\right)^{\frac{3}{2}},$$

and neglected all the rest. Then a pure logarithm results. Our aim in this Appendix was to justify this by showing that the pure logarithm can be approximated as well as we wish by adding suitable terms to $f(m)$.