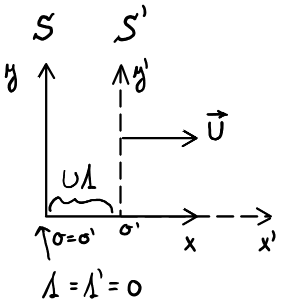
Speciální Teorie Relativity




Speciální Lorentzova Transformace (podél osy x)

$$x' = \frac{x - ut}{\sqrt{1-\frac{u^2}{c^2}}} \qquad t' = \frac{t - \frac{u}{c^2}x}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$y' = y \qquad z' = z \qquad \beta = \frac{u}{c} \qquad c = 3\cdot 10^8\ \mathrm{m\,s^{-1}} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$\longleftrightarrow$ Inverzní Tr.

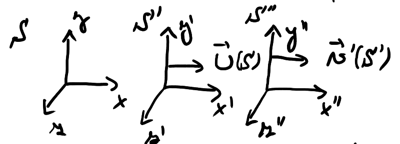
$$x = \frac{x' - u't'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{x' + ut'}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$y = y' \qquad z = z' \qquad \boxed{u' = -u}$$

$$t = \frac{t' - \frac{u'}{c^2}x'}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{t + \frac{u}{c^2}x}{\sqrt{1-\frac{u^2}{c^2}}}$$

35) Odvoďte relativistický zákon skládá rovnoběžných rychlostí složením dvou speciálních Lorentzových transformací. Jsou speciální Lorentzovy transformace podél osy x záměnné – záleží na jejich pořadí?

nezáleží

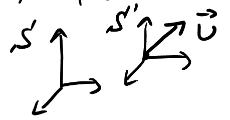


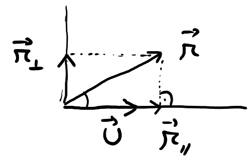

$$x'' = \frac{x' - v't'}{\sqrt{1-\frac{v'^2}{c^2}}} = \frac{\left(\frac{x-ut}{\sqrt{1-\frac{u^2}{c^2}}} - v'\frac{t-\frac{u}{c^2}x}{\sqrt{1-\frac{u^2}{c^2}}}\right)}{\sqrt{1-\frac{v'^2}{c^2}}} = \frac{x\left(1+\frac{uv'}{c^2}\right) - (u+v')t}{\sqrt{1-\frac{v'^2}{c^2}}\sqrt{1-\frac{u^2}{c^2}}} = \frac{x - \frac{(u+v')}{\left(1+\frac{uv'}{c^2}\right)}t}{\sqrt{\frac{1-\frac{v'^2}{c^2}-\frac{u^2}{c^2}+\frac{v'^2u^2}{c^4}+\underbrace{2\frac{uv'}{c^2}-2\frac{uv'}{c^2}}_{0}}{1+2\frac{uv'}{c^2}+\frac{u^2v'^2}{c^4}}}} =$$

$$= \frac{x - \frac{u+v'}{1+\frac{uv'}{c^2}}t}{\sqrt{1-\frac{1}{c^2}\frac{(v'^2+u^2+2v'u)}{\left(1+\frac{uv'}{c^2}\right)^2}}} = \frac{x - \frac{u+v'}{1+\frac{uv'}{c^2}}t}{\sqrt{1-\frac{1}{c^2}\left(\frac{u+v'}{1+\frac{uv'}{c^2}}\right)^2}} \Rightarrow \boxed{v = \frac{u+v'}{1+\frac{uv'}{c^2}}}$$

$u \leftrightarrow v'$

36) Zapište Lorentzovy transformace ve vektorovém tvaru.




$$\vec{r}_\parallel = \frac{\vec{u}\cdot\vec{r}}{|\vec{u}||\vec{r}|}\cdot|\vec{r}|\cdot\frac{\vec{u}}{|\vec{u}|} = \frac{\vec{u}\cdot\vec{r}}{u^2}\vec{u} \qquad \vec{r}_\perp = \vec{r} - \vec{r}_\parallel$$

$$\vec{r}' = \vec{r}'_\parallel + \underbrace{\vec{r}'_\perp}_{=\vec{r}_\perp} = \gamma(\vec{r}_\parallel - \vec{u}t) + \underbrace{\vec{r}-\vec{r}_\parallel}_{\vec{r}_\perp} = (\gamma-1)\vec{r}_\parallel - \gamma\vec{u}t + \vec{r} = \vec{r} - \gamma\vec{u}t + (\gamma-1)\frac{\vec{u}\cdot\vec{r}}{u^2}\vec{u} = \boxed{\vec{r} + \left(\frac{(\gamma-1)}{u^2}\vec{u}\cdot\vec{r} - \gamma t\right)\vec{u}}$$

$$t' = \gamma\left(t - \frac{u}{c^2}r_\parallel\right) = \gamma\left(t - \frac{\vec{u}}{c^2}\vec{r}_\parallel\right) = \boxed{\gamma\left(t - \frac{\vec{u}\cdot\vec{r}}{c^2}\right)}$$

$\vec{u}\cdot\vec{r}_\perp = 0$

37) Najděte matici boostu (speciální Lorentzovy transformace) v libovolném směru.

čtyřvektor (polohy) $x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$ báze $E_4 = \mathbb{R}^4$ $\quad \underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{\vec{e}_1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\underbrace{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\vec{e}_4}$ $\}ct$, $\}\vec{r}$

$\mu \in \{0,1,2,3\}$

$${}^{\varepsilon}A^{\tilde{\varepsilon}} = \left([A\vec{e}_1]_{\tilde{\varepsilon}}, \dots, [A\vec{e}_4]_{\tilde{\varepsilon}}\right) \qquad (\varepsilon = \tilde{\varepsilon})$$

Matice Boostu je symetrická!

$$\vec{r}' = \vec{r} + \left(\frac{(\gamma-1)}{u^2}\vec{u}\cdot\vec{r} - \frac{\gamma}{c}\underbrace{(ct)}_{x^0}\right)\vec{u} \qquad x'^0 = ct' = \gamma\left(\underbrace{ct}_{x^0} - \frac{\vec{u}\cdot\vec{r}}{c}\right)$$

$$\mathbb{A} = \left(\begin{array}{c|c} \gamma & -\frac{\gamma}{c}\vec{u}^T \\ \hline -\frac{\gamma}{c}\vec{u} & \mathbb{I} \end{array}\right) + \left(\begin{array}{c|c} 0 & \vec{0}^T \\ \hline \vec{0} & \vec{u}\cdot\vec{u}^T \end{array}\right)\frac{(\gamma-1)}{u^2}$$

$$\mathbb{A} = \begin{pmatrix} \gamma & -\frac{\gamma}{c}u_1 & -\frac{\gamma}{c}u_2 & -\frac{\gamma}{c}u_3 \\ -\frac{\gamma}{c}u_1 & 1+\frac{(\gamma-1)}{u^2}u_1u_1 & \frac{(\gamma-1)}{u^2}u_2u_1 & \frac{(\gamma-1)}{u^2}u_3u_1 \\ -\frac{\gamma}{c}u_2 & \frac{(\gamma-1)}{u^2}u_1u_2 & 1+\frac{(\gamma-1)}{u^2}u_2u_2 & \frac{(\gamma-1)}{u^2}u_3u_2 \\ -\frac{\gamma}{c}u_3 & \frac{(\gamma-1)}{u^2}u_1u_3 & \frac{(\gamma-1)}{u^2}u_2u_3 & 1+\frac{(\gamma-1)}{u^2}u_3u_3 \end{pmatrix}$$

38) Odvoďte relativistický zákon skládání rychlostí pro libovolnou vzájemnou orientaci obou rychlostí. Jak se zjednoduší pro V<<c. Jaká bude velikost výsledné rychlosti? $(\vec U' = -\vec U)$ $\gamma' = \gamma$

$$t = \gamma'\left(t' - \frac{\vec U'}{c^2}\cdot \vec r'\right) = \gamma\left(t' + \frac{\vec U\cdot\vec r'}{c^2}\right)$$

$$\vec r = \vec r' + \left(\frac{(\gamma'-1)}{U'^2}(\vec U'\cdot\vec r') - \gamma' t'\right)\vec U' = \vec r' + \left(\frac{(\gamma-1)}{U^2}\vec U\cdot\vec r' + \gamma t'\right)\vec U \qquad \vec r = \vec r(t) = \vec r(\vec r'(t'), t')$$

$$\vec v = \frac{d\vec r}{dt} = \frac{d\vec r}{dt'}\cdot\frac{dt'}{dt} = \frac{d\vec r}{dt'}\cdot\frac{1}{\frac{dt}{dt'}} = \left[\frac{d\vec r'}{dt'} + \left(\frac{(\gamma-1)}{U^2}\vec U\cdot\frac{d\vec r'}{dt'} + \gamma\frac{dt'}{dt'}\right)\vec U\right]\frac{1}{\gamma\left(\underbrace{\frac{dt'}{dt'}}_{1} + \frac{\vec U}{c^2}\cdot\underbrace{\frac{d\vec r'}{dt'}}_{\vec v'}\right)} =$$

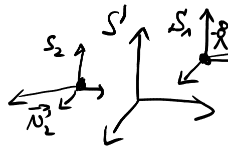
$$= \frac{\vec v' + \left(\frac{(\gamma-1)}{U^2}\vec U\cdot\vec v' + \gamma\right)\vec U}{\gamma\left(1 + \frac{\vec U\cdot\vec v'}{c^2}\right)} = \frac{\vec U + \vec v'\sqrt{1-\frac{U^2}{c^2}} + \left(1-\sqrt{1-\frac{U^2}{c^2}}\right)\frac{\vec U\cdot\vec v'}{U^2}\vec U}{1 + \frac{\vec U\cdot\vec v'}{c^2}}$$

$$U << c \rightarrow \frac{\vec U + \vec v'}{1}$$

$$\frac{U}{c} \sim 0$$

$$\frac{(\gamma-1)}{\gamma} = 1 - \frac{1}{\gamma}$$

39) Rerlativní rychlost dvou částic je definována jako rychlost jedné z nich v soustavě, v níž je druhá částice v klidu. Určete kvadrát $v_{rel}^2$, jestliže v některé inerciální soustavě mají částice rychlosti $\vec v_1, \vec v_2$.




$$v_{rel} = |\vec v_1(S_2)| = |\vec v_2(S_1)| \qquad \vec U = -\vec v_1 \qquad \vec v' = \vec v_2$$

$$|\vec v| = \sqrt{\vec v\cdot\vec v} = \frac{1}{1+\frac{\vec U\cdot\vec v'}{c^2}}\sqrt{\vec U^2 + 2\vec U\cdot\vec v'\sqrt{1-\frac{U^2}{c^2}} + 2\left(1-\sqrt{1-\frac{U^2}{c^2}}\right)\vec U\cdot\vec v' + \vec v'^2\left(1-\frac{U^2}{c^2}\right) + 2\left(\sqrt{1-\frac{U^2}{c^2}} - 1 + \frac{U^2}{c^2}\right)\frac{(\vec U\cdot\vec v')^2}{U^2} + \left(1-\sqrt{1-\frac{U^2}{c^2}}\right)^2\frac{(\vec U\cdot\vec v')^2}{U^2}}$$

$$= \frac{1}{1+\frac{\vec U\cdot\vec v'}{c^2}}\sqrt{\vec U^2 + 2\vec U\cdot\vec v' + \vec v'^2 - \frac{U^2 v'^2}{c^2} - 2\frac{(\vec U\cdot\vec v')^2}{U^2} + 2\frac{(\vec U\cdot\vec v')^2}{c^2} + \cancel{2\sqrt{1-\frac{U^2}{c^2}}\frac{(\vec U\cdot\vec v')^2}{U^2}} + \frac{(\vec U\cdot\vec v')^2}{U^2} - \cancel{2\sqrt{1-\frac{U^2}{c^2}}\frac{(\vec U\cdot\vec v')^2}{U^2}} + \left(1-\frac{U^2}{c^2}\right)\frac{(\vec U\cdot\vec v')^2}{U^2}}$$

$$= \frac{1}{1+\frac{\vec U\cdot\vec v'}{c^2}}\sqrt{(\vec U+\vec v')^2 - \frac{U^2 v'^2}{c^2} + \frac{(\vec U\cdot\vec v')^2}{c^2}} = \frac{1}{1+\frac{\vec U\cdot\vec v'}{c^2}}\sqrt{(\vec U+\vec v')^2 - \frac{1}{c^2}(\vec U\times\vec v')^2} \qquad v_{rel}^2 = \frac{(\vec v_2-\vec v_1)^2 - \frac{1}{c^2}(-\vec v_1\times\vec v_2)^2}{\left(1-\frac{\vec v_1\cdot\vec v_2}{c^2}\right)^2}$$

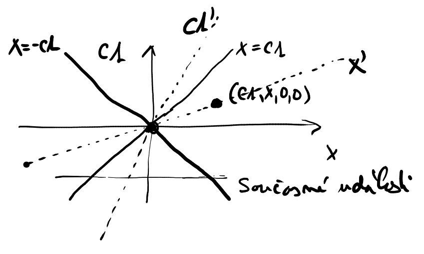
$$(Uv'\cos\varphi)^2 - U^2v'^2 = -U^2v'^2\sin^2\varphi = -|\vec U\times\vec v'|^2$$

40) Relativita současnosti a soumístnosti. Uvažujte dvě inerciální soustavy S a S' spojené speciální Lorentzovou transformací. Určete rychlost U soustavy S' vůči soustavě S tak aby

a) událost o souřadnicích (ct, x, 0, 0) splňujících $(ct)^2 - x^2 = \Delta s^2 < 0$ v soustavě S byla v soustavě S' současná s událostí o souřadnicích (0, 0, 0, 0) $\rightarrow \boxed{(0, x', 0, 0)}$

b) událost o souřadnicích (ct, x, 0, 0) splňujících $(ct)^2 - x^2 = \Delta s^2 > 0$ v soustavě S byla v soustavě S' soumístná s událostí o souřadnicích (0, 0, 0, 0) $\rightarrow (ct', 0, 0, 0)$

a)



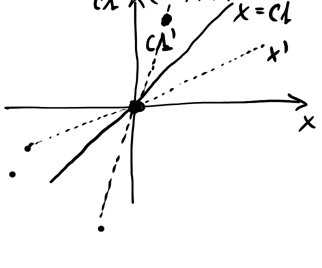

$$x = \frac{x' + Ut'}{\sqrt{1-\frac{U^2}{c^2}}} = \frac{x' + 0}{\sqrt{1-\frac{U^2}{c^2}}} \Rightarrow x' = \sqrt{1-\frac{U^2}{c^2}}\,x$$

$$0 > \Delta s^2 = (ct)^2 - x^2 = \hat x^2$$

$$x^2 > (ct)^2 \qquad 1 > \left(\frac{ct}{x}\right)^2 \qquad \boxed{1 > \frac{ct}{x}}$$

$$t = \frac{t' + \frac{Ux'}{c^2}}{\sqrt{1-\frac{U^2}{c^2}}} = \frac{0 + \frac{Ux'}{c^2}}{\sqrt{1-\frac{U^2}{c^2}}} \qquad t = \frac{U\sqrt{1-\frac{U^2}{c^2}}\,x}{c^2\sqrt{1-\frac{U^2}{c^2}}} = \frac{Ux}{c^2} \Rightarrow U = c^2\frac{t}{x} < c$$

b)




$$x = \gamma(x' + Ut') = \gamma Ut' = \gamma U\frac{t}{\gamma} = Ut \Rightarrow \underline{U = \frac{x}{t} < c}$$

$$t = \gamma\left(t' + \frac{Ux'}{c^2}\right) = \gamma t' \Rightarrow t' = \frac{t}{\gamma}$$

$$(ct', 0, 0, 0)$$

$$0 < \Delta s^2 = (ct)^2 - x^2$$

$$(ct)^2 > x^2$$

$$1 > \left(\frac{x}{ct}\right)^2 \qquad c > \frac{x}{t}$$