(b) $\vec{A}=(0,Bx_1,0)$

$L=\frac{1}{2}m\dot{x}_i^2+eBx_1\dot{x}_2$

cyklické

$\frac{\partial L}{\partial t}=0 \Rightarrow E=\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L = \underbrace{\frac{1}{2}m\dot{x}_i^2}=\text{konst}$

$\frac{\partial L}{\partial x_2}=0 \Rightarrow p_2=\frac{\partial L}{\partial \dot{x}_2}=\underbrace{m\dot{x}_2+eBx_1}=\text{konst.}$

$\frac{\partial L}{\partial x_3}=0 \Rightarrow p_3=\frac{\partial L}{\partial \dot{x}_3}=\underbrace{m\dot{x}_3}=\text{konst.}$

$\Big\}$ 3 I.P.

Celkem

(a) $E,\ p_3,\ L_3$

(b) $E,\ p_3,\ p_2$

$\to$ 4 I.P.

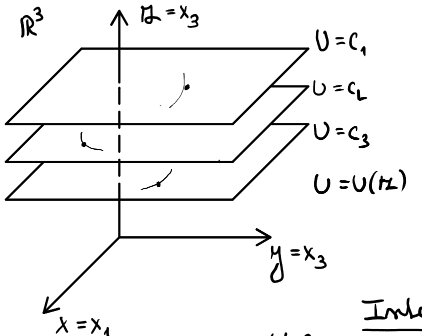
(c) $E,\ p_1,\ p_3$ $\swarrow$ 5 I.P.

$\Delta=3$ max $2\Delta=6$ I.P.

$2\Delta-1$ I.P. ve kterých není t $\to$ trajektorie

62. Které složky celkové hybnosti a celkového momentu hybnosti soustavy částic se zachovávají v silovém poli, jehož ekvipotenciální plochy jsou

a) roviny kolmé k ose z



$\vec{x}_\alpha=\begin{pmatrix}x_\alpha\\y_\alpha\\z_\alpha\end{pmatrix}=\begin{pmatrix}x_{\alpha_1}\\x_{\alpha_2}\\x_{\alpha_3}\end{pmatrix}$ $\quad L=\overbrace{\sum_{\alpha=1}^{N}\frac{1}{2}m_\alpha\dot{x}_{\alpha_i}^2}^{T}-U_{INT}(|\vec{x}_\alpha-\vec{x}_\beta|)-U(x_{\alpha_3})$ $\quad \frac{\partial L}{\partial t}=0\Rightarrow E=T+U_{INT}+U$

Interakční potenciál

Teorém Noetherové

Translace $\quad x'_{\alpha_i}=x_{\alpha_i}+\varepsilon a_i \quad Y_{\alpha_i}=\left(\frac{\partial x'_{\alpha_i}}{\partial\varepsilon}\right)_{\varepsilon=0}=a_i$

$\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\0\end{pmatrix} \quad \dot{x}'_{\alpha_i}=\dot{x}_{\alpha_i}+0$

Invariance $L(\vec{x}',\dot{\vec{x}}',t)=L(\vec{x},\dot{\vec{x}},t)$

$L=\sum_\alpha\frac{1}{2}m_\alpha\dot{x}_{\alpha_i}'^2-U_{INT}(|\vec{x}'_\alpha-\vec{x}'_\beta|)-U(x'_{\alpha_3})=$

$=\sum_\alpha\frac{1}{2}m_\alpha\dot{x}_{\alpha_i}^2-U_{INT}(|\vec{x}_\alpha+\cancel{\varepsilon\vec{a}}-\vec{x}_\beta-\cancel{\varepsilon\vec{a}}|)$

Integrál Pohybu

$I=\sum_{\alpha=1}^{N}\sum_{j=1}^{3}\left(\frac{\partial L}{\partial\dot{x}_{\alpha_j}}\right)\left(\frac{\partial x'_{\alpha_j}}{\partial\varepsilon}\right)_{\varepsilon=0}=\sum_{\alpha=1}^{N}\sum_{j=1}^{3}m_\alpha\dot{x}_{\alpha_j}\cdot Y_j=\sum_{\alpha=1}^{N}m_\alpha\dot{x}_{\alpha_j}a_j=\underbrace{\left(\sum_{\alpha=1}^{N}m_\alpha\dot{x}_{\alpha_j}\right)}_{P_j}a_j \quad -U(x_{\alpha_3}+\underbrace{\varepsilon a_3}_{0})\Rightarrow\boxed{a_3=0}$

$=P_ja_j=P_1a_1+P_2a_2+0=\underline{P_1a_1+P_2a_2}$ $\quad$ pro $a_2=0\Rightarrow P_1$, $a_1=0\Rightarrow P_2$ $\Big\}$ I.P.

Rotace kolem osy x_3

$x'_{\alpha_1}=x_{\alpha_1}\cos\varepsilon-x_{\alpha_2}\sin\varepsilon \qquad Y_{\alpha_1}=\left(\frac{\partial x'_{\alpha_1}}{\partial\varepsilon}\right)_{\varepsilon=0}=-x_{\alpha_2}$

$x'_{\alpha_2}=x_{\alpha_1}\sin\varepsilon+x_{\alpha_2}\cos\varepsilon \qquad Y_{\alpha_2}=\left(\frac{\partial x'_{\alpha_2}}{\partial\varepsilon}\right)_{\varepsilon=0}=x_{\alpha_1}$

$x'_{\alpha_3}=x_{\alpha_3} \qquad Y_{\alpha_3}=\ \dots\ =0$

Invariance L

$T=\sum_\alpha\frac{1}{2}m_\alpha\underline{\vec{v}_\alpha^2}$ ✓

$U_{INT}=U_{INT}(|\vec{x}_\alpha-\vec{x}_\beta|)$ ✓

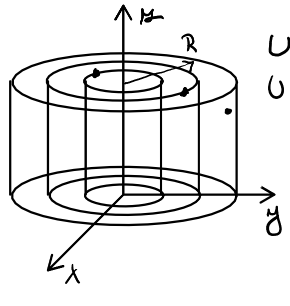
$U(x'_{\alpha_3})=U(x_{\alpha_3})$ ✓

Integrál Pohybu

$I=\sum_{\alpha=1}^{N}\left(\frac{\partial L}{\partial\dot{x}'_{\alpha_j}}\right)\left(\frac{\partial x'_{\alpha_j}}{\partial\varepsilon}\right)_{\varepsilon=0}=\sum_{\alpha=1}^{N}m_\alpha\dot{x}_{\alpha_j}\underbrace{Y_{\alpha_j}}_{Y_{\alpha_j}}=\sum_{\alpha=1}^{N}\underline{m_\alpha}(\underline{\dot{x}_{\alpha_1}}(-x_{\alpha_2})+\dot{x}_{\alpha_2}(x_{\alpha_1})+0)=\sum_{\alpha=1}^{N}x_{\alpha_1}p_{\alpha_2}-x_{\alpha_2}p_{\alpha_1}=\sum_{\alpha=1}^{N}l_{\alpha_3}=\underline{L_3}$

celk. moment hybnosti

b) válcové plochy s osou z



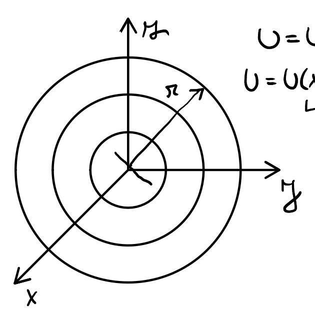
$U=U(R)$

$U=U(\underbrace{x^2+y^2})$

Translace podél $z\Rightarrow P_3$

Rotace okolo $z\Rightarrow L_3$

c) kulové plochy se středem v počátku



$U=U(r)$

$U=U(\underbrace{x^2+y^2+z^2})$

Translace X

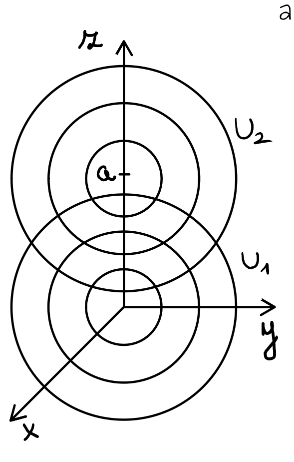
Rotace – lib. I.P.

okolo $x\Rightarrow L_1$

$y\Rightarrow L_2$

$z\Rightarrow L_3$

d) $U=U_1+U_2$ kde U_1 a U_2 jsou jako v c) ale s posunutými středy



Rotace kolem osy z

$\Downarrow$

L_3

59. Nechť má Lagrangeova funkce tvar $L = \frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2) - U\big((\underbrace{x_1^2+x_2^2}_{\xi}), x_3\big)$ Ukažte, že vektorové pole $\vec{Y} = \begin{pmatrix}-x_2\\ x_1\\ 0\end{pmatrix}$ splňuje rovnici $\sum_{j=1}^{n}\left[\frac{\partial L}{\partial x_j}Y_j + \frac{\partial L}{\partial \dot{x}_j}\underbrace{\left(\sum_{k=1}^{n}\frac{\partial Y_j}{\partial x_k}\dot{x}_k + \frac{\partial Y_j}{\partial t}\right)}_{\dot{Y}_j}\right] = 0$ a najděte odpovídající zachovávající se veličinu.

Infinitesimální transformace (ε malé)

$x_i' = x_i + \varepsilon Y_i$ $\qquad x_1' = x_1 - \varepsilon x_2$

$\dot{x}_i' = \dot{x}_i + \varepsilon \dot{Y}_i$ $\qquad x_2' = x_2 + \varepsilon x_1$

$x_3' = x_3$

Invariance L $\qquad L'(\vec{x},\dot{\vec{x}},\varepsilon) := L(\vec{x}'(\vec{x},\varepsilon), \dot{\vec{x}}'(\vec{x},\dot{\vec{x}},\varepsilon)) \overset{!}{=} L(\vec{x},\dot{\vec{x}})$ do 1. řádu v ε

$$0 = \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial L}{\partial x_j}\underbrace{\left(\frac{\partial x_j'}{\partial \varepsilon}\right)_{\varepsilon=0}}_{Y_j} + \left(\frac{\partial L}{\partial \dot{x}_j}\right)\left(\frac{\partial \dot{x}_j'}{\partial \varepsilon}\right)_{\varepsilon=0} = \frac{\partial L}{\partial x_j}Y_j + \left(\frac{\partial L}{\partial \dot{x}_j}\right)\frac{d}{dt}\underbrace{\left(\frac{\partial x_j'}{\partial \varepsilon}\right)_{\varepsilon=0}}_{Y_j} = \frac{\partial L}{\partial x_j}Y_j + \frac{\partial L}{\partial \dot{x}_j}\dot{Y}_j$$

$$-\frac{\partial U}{\partial \xi}\cdot\frac{\partial \xi}{\partial x_1}(-x_2) + \left(-\frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x_2}\right)(x_1) + \frac{\partial U}{\partial x_3}\cdot(0) + \cancel{m\dot{x}_1(-\dot{x}_2)} + \cancel{m\dot{x}_2(\dot{x}_1)} + m\dot{x}_3\, 0 = -\frac{\partial U}{\partial \xi}\left(2x_1(-x_2) + 2x_2(x_1)\right) = \underline{\underline{0}}\ \checkmark$$

Integrál pohybu $\qquad I = \sum_{j=1}^{3}\left(\frac{\partial L}{\partial \dot{x}_j}\right)Y_j = m\dot{x}_1(-x_2) + m\dot{x}_2(x_1) + \cancel{m\dot{x}_3 0} = L_3$

62. Nechť má Lagrangeova funkce tvar $L = \frac{1}{2}m\dot{x}^2$ Ukažte, že pohybové rovnice (LR2D) jsou invariantní vůči transformaci (nazývané škálování) $x \mapsto x' = x e^{\varepsilon}$ $\quad \varepsilon \in \mathbb{R}$

Jak vypadá vektorové pole $\vec{Y}$ pro tuto grupu transformací? Je veličina $F(x,\dot{x},t) = \sum_{j=1}^{n} Y_j(x,t)\frac{\partial L}{\partial \dot{x}_j}(x,\dot{x},t)$ v tomto případě integrálem pohybu?

Transformace $\qquad \phi^0(x) = x\cdot e^0 = x$ ✓

$x' = e^{\varepsilon}x = \phi^{\varepsilon}(x) \qquad \phi^{\varepsilon}\circ\phi^{\delta}(x) = (xe^{\delta})e^{\varepsilon} = xe^{\varepsilon+\delta} = \phi^{\varepsilon+\delta}(x)$ ✓

$Y = \left(\frac{\partial x'}{\partial \varepsilon}\right)_{\varepsilon=0} = xe^{\varepsilon}\big|_{\varepsilon=0} = x$

Lagr. fce

$L(\dot{x}') = \frac{1}{2}m\dot{x}'^2 = \frac{1}{2}m\dot{x}^2\underbrace{e^{2\varepsilon}}_{\neq 1} \neq L(\dot{x}) \quad \forall\varepsilon$

není invariantní

Pohyb. rce $\quad \underline{m\ddot{x} = 0}$ je invariantní

$\dot{x}' = \dot{x}e^{\varepsilon}$

$\ddot{x}' = \ddot{x}e^{\varepsilon}$ $\qquad \underline{0 = m\ddot{x}'} = \underline{m\ddot{x}e^{\varepsilon}}\ /\frac{1}{e^{\varepsilon}} \to \underline{m\ddot{x} = 0}$

Fce $\quad F = m\dot{x}\cdot x \quad$ je I.P.? $\quad 0 = \frac{dF}{dt} = m\ddot{x}x + m\dot{x}^2 = 0 + m\dot{x}^2 \neq 0$

není I.P.!

63. Ukažte, že při pohybu částice v poli $U = \frac{\alpha}{r}$ kde $\alpha \neq 0$ existuje vektorový integrál pohybu $\vec{A} = \vec{v}\times\vec{L} + \alpha\frac{\vec{r}}{r}$ nazývaný Runge–Lenzův vektor, který je specifický právě pro toto pole.

$L = \frac{1}{2}m\vec{v}^2 - \frac{\alpha}{r} = \frac{1}{2}m\dot{x}_i^2 - \frac{\alpha}{\sqrt{\sum_i x_i^2}} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) - \frac{\partial L}{\partial x_j} = 0 \qquad \frac{d}{dt}(m\dot{x}_j) - \frac{\partial}{\partial x_j}\left(-\frac{\alpha}{r}\right) = m\ddot{x}_j - \alpha\left(+\frac{1}{r^2}\right)\frac{x_j}{r} = 0 \Rightarrow \boxed{m\ddot{\vec{r}} = \alpha\frac{\vec{r}}{r^3}}$

$r = \sqrt{\sum_j x_j^2}$

$\alpha\frac{\partial\left(\frac{1}{r}\right)}{\partial r}\frac{\partial r}{\partial x_j}$

$$\frac{d\vec{A}}{dt}\Big|_{\text{podél trajektorie}} = \dot{\vec{v}}\times\vec{L} + \vec{v}\times\dot{\vec{L}} + \alpha\frac{\dot{\vec{r}}r - \vec{r}\dot{r}}{r^2} = \ddot{\vec{r}}\times(\vec{r}\times m\dot{\vec{r}}) + \dot{\vec{r}}\times(\underbrace{\cancel{\dot{\vec{r}}\times m\dot{\vec{r}}}}_{0} + \vec{r}\times m\ddot{\vec{r}}) + \alpha\frac{\dot{\vec{r}}r - \vec{r}\frac{\vec{r}\cdot\dot{\vec{r}}}{r}}{r^2} = \vec{r}\left(\alpha\frac{\vec{r}}{r^3}\cdot\dot{\vec{r}}\right) - \dot{\vec{r}}\left(\alpha\frac{\vec{r}}{r^3}\cdot\vec{r}\right) + \alpha\frac{\dot{\vec{r}}r}{r^2} - \alpha\vec{r}\left(\frac{\vec{r}\cdot\dot{\vec{r}}}{r^3}\right) = \underline{\underline{0}}$$

$\vec{r}\times\alpha\frac{\vec{r}}{r^3} = 0$

$m\left(\vec{r}(\dot{\vec{r}}\cdot\ddot{\vec{r}}) - \dot{\vec{r}}(\vec{r}\cdot\ddot{\vec{r}})\right)$

$\dot{r} = \frac{\partial r}{\partial x_j}\dot{x}_j = \frac{x_j}{r}\dot{x}_j = \frac{\vec{r}\cdot\dot{\vec{r}}}{r}$

64. Užitím integrálů pohybu najděte tvar trajektorie částice v poli $U = \frac{\alpha}{r}$.

$L = \underbrace{\frac{1}{2}m\dot{x}_i^2}_{T} - \underbrace{\frac{\alpha}{\sqrt{x_i^2}}}_{U} \qquad \frac{\partial L}{\partial t} = 0 \Rightarrow E = T + U$

cyklický nepřímo

Rotace kolem všech os $\Rightarrow \vec{L}$ moment hybnosti

$U = \frac{\alpha}{r} \Rightarrow \vec{A}$ Runge-Lenz

$\Big\}$ I.P. $\to$ 7

$n = 3 \quad 2n = 6$

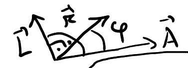
① $\underline{x_i L_i} = \vec{r}\cdot\vec{L} = \vec{r}\cdot(\vec{r}\times m\dot{\vec{r}}) = \underline{0} \Rightarrow \vec{r}\perp\vec{L}$ leží v rovině jdoucí počátkem

konst.

Rce roviny $\qquad$ (kolmé na $\vec{L}$)

③ Zvolíme souřadnice v rovině

$r = |\vec{r}| \quad \varphi \measuredangle \vec{r}\,a\,\vec{A}$



② $\underline{A_i L_i} = \vec{A}\cdot\vec{L} = (\vec{v}\times\vec{L})\cdot\vec{L} + \alpha\frac{\vec{r}}{r}\cdot\vec{L} = 0 + 0 = \underline{0} \Rightarrow \vec{A}\perp\vec{L}$

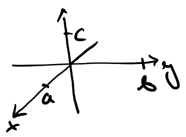
Tvar Trajektorie

$$\boxed{r = \frac{L^2}{m(A\cos\varphi - \alpha)}}$$

③ $|\vec{A}||\vec{r}|\cos\varphi = \vec{A}\cdot\vec{r} = \underbrace{(\vec{v}\times\vec{L})\cdot\vec{r}}_{\varepsilon_{ijk}v_j L_k r_i \,=\, L_k \varepsilon_{kij} r_i v_j \frac{m}{m}} + \alpha\frac{\vec{r}}{r}\cdot\vec{r} = \frac{1}{m}L_k^2 + \alpha r = \frac{1}{m}\vec{L}^2 + \alpha r$

r

69. Odvoďte podmínky pro reálná a virtuální posunutí pro hmotný bod pohybující se po trojosém elipsoidu, jehož délky poloos jsou závislé na čase.



$$f(x,y,z,t) = \frac{x^2}{a(t)^2} + \frac{y^2}{b^2(t)} + \frac{z^2}{c(t)^2} - 1 = 0$$

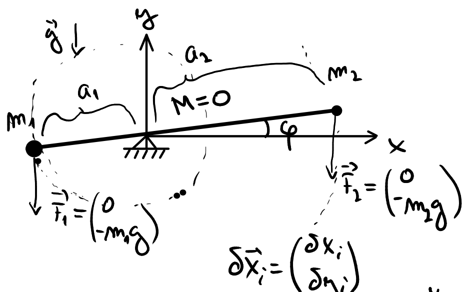
Reálná posunutí $0 = df = \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial t} dt$

$$0 = df = \underbrace{\frac{2x}{a^2} dx}_{} + \underbrace{\frac{2y}{b^2} dy}_{} + \underbrace{\frac{2z}{c^2} dz}_{} - 2\left(\frac{x^2}{a^3}\dot{a} + \frac{y^2}{b^3}\dot{b} + \frac{z^2}{c^3}\dot{c}\right) dt$$

Virtuální posunutí

$$0 = \delta f = \frac{\partial f}{\partial x_i} \delta x_i \qquad 0 = \delta f = \underbrace{\frac{2x}{a^2}\delta x}_{} + \underbrace{\frac{2y}{b^2}\delta y}_{} + \underbrace{\frac{2z}{c^2}\delta z}_{}$$

Př. Pomocí principu virtuální práce odvoďte podmínku rovnováhy na páce.



$$\delta\vec{x}_i = \begin{pmatrix}\delta x_i \\ \delta y_i\end{pmatrix}$$

$$\delta A = \vec{F} \cdot \delta\vec{x}$$

akční síla ($\mathbb{R}^{3N}$)

Vazby (Ideální) $(\lambda_i \nabla f_i \cdot \delta\vec{x} = 0)$

$$x_1^2 + y_1^2 - a_1^2 = 0$$
$$x_2^2 + y_2^2 - a_2^2 = 0$$
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 - (a_1 + a_2)^2 = 0$$

$\nu = 2 \cdot 2 - 3 = 1$, φ

$$0 = \delta A = \vec{F}_1 \cdot \delta\vec{x}_1 + \vec{F}_2 \cdot \delta\vec{x}_2 = -m_1 g \underbrace{\delta y_1} - m_2 g \underbrace{\delta y_2} = -g\underbrace{(-m_1 a_1 + m_2 a_2)\cos\varphi}_{=0}\underbrace{\delta\varphi} = 0$$

závislé

$\forall \delta\varphi$

$x_1 = -a_1 \cos\varphi$

$y_1 = -a_1 \sin\varphi \qquad \delta y_1 = -a_1 \cos\varphi\, \delta\varphi$

$x_2 = a_2 \cos\varphi$

$y_2 = a_2 \sin\varphi \qquad \delta y_2 = a_2 \cos\varphi\, \delta\varphi$

$$\boxed{m_1 a_1 = m_2 a_2}$$

$\cos\varphi = 0$

$$\boxed{\varphi = \frac{\pi}{2}, \frac{3}{2}\pi}$$