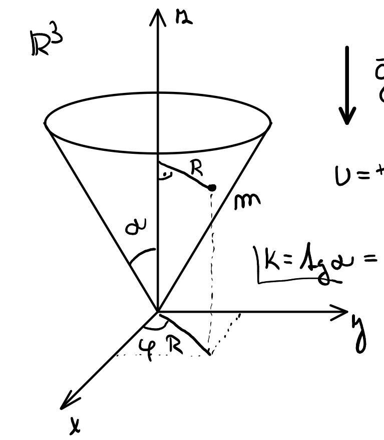
47. Hmotný bod m v homogennním tíhovém poli klouže bez tření po kruhovém kuželi svisle stojícím na špici. Odvoďte pohybové rovnice v cylindrických souřadnicích a příslušné zákony zachování. Diskutujte případ obecné holonomní vazby na rotační plochu R=R(z).



$\downarrow \vec{g}$

$U = +mgz$

$\boxed{K = \operatorname{tg}\alpha = \frac{R}{z}} \Rightarrow Kz = R$

1) osy

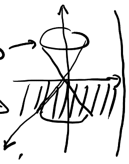
2) $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$

3) Vazby $\quad f(x,y,z) = \sqrt{x^2+y^2} - Kz = 0$

$\boxed{s = 3\cdot 1 - 1 = 2}$

$f(x,y,z) = x^2+y^2-K^2z^2 = 0$

$z \geq 0$



Holonomní, Skleronomní, Udržující, Ideální

(Jinak φ, R)

4) obecné souřadnice φ, z

cylindrické $\to$ Vazba $\to$ obecné s.

$x = R\cos\varphi$
$y = R\sin\varphi$
$z = z$

$\boxed{R - Kz = 0}$

$R = Kz$

$x = Kz\cos\varphi$
$y = Kz\sin\varphi$
$z = z$

$\dot{x} = K\dot{z}\cos\varphi - Kz\dot{\varphi}\sin\varphi$
$\dot{y} = K\dot{z}\sin\varphi + Kz\dot{\varphi}\cos\varphi$
$\dot{z} = \dot{z}$

(5) $L = \frac{1}{2}m(K^2\dot{z}^2 + K^2z^2\dot{\varphi}^2 + \dot{z}^2) - mgz$

(6) $\boxed{s=2}$ $\quad \varphi: \quad \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{\varphi}}\right)}_{p_\varphi} - \underbrace{\frac{\partial L}{\partial \varphi}}_{=0} = \frac{d}{dt}\underbrace{\left(mK^2z^2\dot{\varphi}\right)}_{p_\varphi \text{ je I.P.}} - 0 = 0 \Rightarrow p_\varphi = mK^2z^2\dot{\varphi} = C \text{ konst.} \to \dot{\varphi} = \frac{C}{mK^2z^2}$

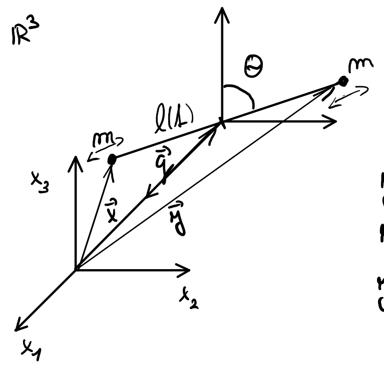
$z: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = \frac{d}{dt}\left(m(1+K^2)\dot{z}\right) - \left(mK^2z\dot{\varphi}^2 - mg\right) = m(1+K^2)\ddot{z} - mK^2z\dot{\varphi}^2 + mg = 0$

$(1+K^2)\ddot{z} - \frac{K^2 z C^2}{m^2K^4z^4} + g = 0 = \ddot{z} - \frac{C^2}{m^2K^2(1+K^2)z^3} + \frac{g}{1+K^2}$

OBECNĚJI $R = R(z)$ - rotační plocha $s = 2$ obecné

$x = R(z)\cos\varphi$
$y = R(z)\sin\varphi$
$z = z$

50. Dva stejné homtné body spojené nehmotnou tyčkou měnící se délky l(t). Najděte obecné souřadnice. Ověřte pravidlo krácení teček a určete vazbové síly.



$\boxed{s = 2\cdot 3 - 1 = 5}$ $\quad$ 1 Vazba $\quad f(\vec{x},\vec{y},t) = (\vec{x}-\vec{y})^2 - l^2(t) = 0 \quad$ H, Rheonomní, I, U

$\sum (x_i - y_i)^2 - l^2(t) = 0$

obecné souřadnice $\vec{q}, \Theta, \varphi$ $\quad$ JINAK $(\vec{x}, \Theta, \varphi)$

$y_1 = q_1 + \frac{l}{2}\sin\Theta\cos\varphi$
$y_2 = q_2 + \frac{l}{2}\sin\Theta\sin\varphi$
$y_3 = q_3 + \frac{l}{2}\cos\Theta$

$x_1 = q_1 - \frac{l}{2}\sin\Theta\cos\varphi$
$x_2 = q_2 - \frac{l}{2}\sin\Theta\sin\varphi$
$x_3 = q_3 - \frac{l}{2}\cos\Theta$

$\dot{x}_1 = \dot{q}_1 - \frac{\dot{l}}{2}\sin\Theta\cos\varphi - \frac{l}{2}\dot{\Theta}\cos\Theta\cos\varphi + \frac{l}{2}\dot{\varphi}\sin\Theta\sin\varphi$
$\dot{x}_2 = \dot{q}_2 - \frac{\dot{l}}{2}\sin\Theta\sin\varphi - \frac{l}{2}\dot{\Theta}\cos\Theta\sin\varphi - \frac{l}{2}\dot{\varphi}\sin\Theta\cos\varphi$
$\dot{x}_3 = \dot{q}_3 - \frac{\dot{l}}{2}\cos\Theta + \frac{l}{2}\dot{\Theta}\sin\Theta$

$\wedge$

$\dot{y}_i = \ldots \quad \forall i = 1,2,3$

Pravidlo krácení teček

$\hat{x}_i = \hat{x}_i(\vec{q}, t) \qquad \dot{\hat{x}}_i = \underbrace{\frac{\partial \hat{x}_i}{\partial q_\gamma}}\dot{q}_\gamma + \underbrace{\frac{\partial \hat{x}_i}{\partial t}}$

$$\boxed{\frac{\partial \dot{\hat{x}}_i}{\partial \dot{q}_\gamma} = \frac{\partial \hat{x}_i}{\partial q_\gamma}}$$

$\frac{\partial \dot{x}_1}{\partial \dot{q}_1} = 1 = \frac{\partial x_1}{\partial q_1} = 1$ ✓ $\qquad \frac{\partial \dot{x}_1}{\partial \dot{\Theta}} = -\frac{l}{2}\cos\Theta\cos\varphi = \frac{\partial x_1}{\partial \Theta} = -\frac{l}{2}\cos\Theta\cos\varphi$ ✓

$\frac{\partial \dot{x}_1}{\partial \dot{q}_2} = 0 = \frac{\partial x_2}{\partial q_2} = 0$ ✓

Vazbové síly $(\mathbb{R}^{2\cdot 3})$

$$\vec{F}^{(Vaz)} = \lambda \cdot \nabla_{(x,y)} f = \lambda \begin{pmatrix} 2(x_1-y_1) \\ 2(x_2-y_2) \\ 2(x_3-y_3) \\ \hline -2(x_1-y_1) \\ -2(x_2-y_2) \\ -2(x_3-y_3) \end{pmatrix} = \begin{pmatrix} \vec{F}^{(Vaz)}_{(\vec{x})} \\ \hline \vec{F}^{(Vaz)}_{(\vec{y})} \end{pmatrix}$$

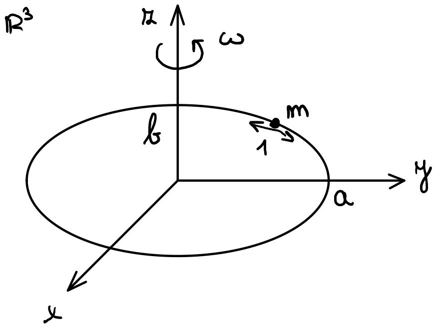
$\vec{F}^{(Vaz)}_{(\vec{x})} = \lambda\frac{\partial f}{\partial \vec{x}} = -\vec{F}^{(Vaz)}_{(\vec{y})} = -\lambda\frac{\partial f}{\partial \vec{y}}$

Konf. Pr.

$M = \mathbb{R}^3 \times S^2$

Sféra

45. Najděte obecné souřadnice pro hmotný bod vázaný na elipsu, která rotuje kolem své vedlejší poloosy s konstantní rychlostí ω.

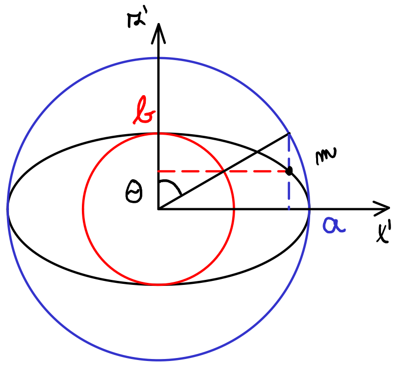


Počet vazeb = ?

Konf. pr. M = rotující elipsa

$\dim M = 1 \Rightarrow \Delta = 1$

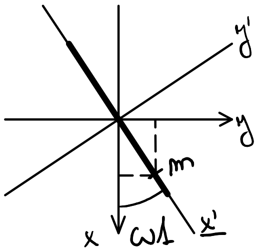
obecná souřadnice - úhel θ



$x' = a\sin\theta$

$y' = 0$

$z' = b\cos\theta$



$$\boxed{\begin{aligned} x &= x'\cos\omega t = a\sin\theta\cos\omega t \\ y &= x'\sin\omega t = a\sin\theta\sin\omega t \\ z &= z' = b\cos\theta \end{aligned}}$$

Jinak otočením

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ \cancel{y'} \\ z' \end{pmatrix} = \begin{pmatrix} x'\cos\omega t \\ x'\sin\omega t \\ z' \end{pmatrix}$$

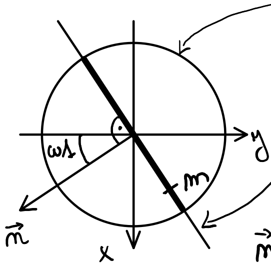
$y' = 0$

Jinak – zadané vazby: 2 vazby

$$f_1(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0$$ Rotační elipsoid H, S, I, U

Poloosy (a, a, b)

$$f_2(x,y,t) = x\sin\omega t - y\cos\omega t = 0$$ Rotující Rovina H, R, I, U



$$\vec{m} = \begin{pmatrix} \sin\omega t \\ -\cos\omega t \\ 0 \end{pmatrix}$$

$\Delta = 3\cdot 1 - 2 = 1$

obecná souřadnice θ

Parametrizujeme f_1

$$\begin{aligned} x &= a\sin\theta\cos\varphi \\ y &= a\sin\theta\sin\varphi \\ z &= b\cos\theta \end{aligned}$$

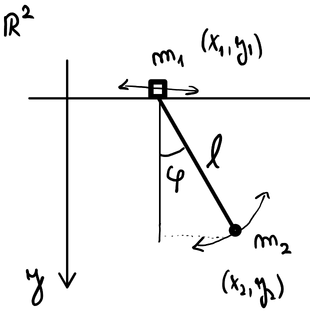
$\underline{a\sin\theta}\cos\varphi\sin\omega t - \underline{a\sin\theta}\sin\varphi\cos\omega t = 0 \quad \forall\theta$

$\cos\varphi\sin\omega t - \sin\varphi\cos\omega t = 0$

$\sin(\omega t - \varphi) = 0 \Rightarrow \varphi = \omega t + k\pi \quad k \in \mathbb{Z}$

vybereme $k = 0 \Rightarrow \boxed{\varphi = \omega t}$

48. Ve vodorovné rovině může klouzat bez tření těleso hmotnosti m1. To je spojeno nehmotnou tyčí délky l s tělesem hmotnosti m2, které koná působením tíže kmitavý pohyb ve svislé rovině. Dokažte, že těleso m2 se pohybuje po elipse a vypočítejte dobu kmitu T tohoto eliptického kyvadla pro malé amplitudy.



$U = -mgy$

$L = T - U$

1, osy

2, $L = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + mgy_1 + m_2 g y_2$

3, Vazby $f_1(y_1) = y_1 = 0$

$f_2(x_1, x_2, y_1, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 - \ell^2 = 0$ } H, S, U, I

Konf. Pr. $M = \mathbb{R} \times S^1$ ← kružnica

$\boxed{s = 2\cdot 2 - 2 = 2}$

4, obecné souřadnice x_1, φ

$x_1 = x_1$ $\quad \dot{x}_1 = \dot{x}_1$ $\quad x_2 = x_1 + \ell\sin\varphi$ $\quad \dot{x}_2 = \dot{x}_1 + \ell\dot{\varphi}\cos\varphi$

$y_1 = 0$ $\quad \dot{y}_1 = 0$ $\quad y_2 = \ell\cos\varphi$ $\quad \dot{y}_2 = -\ell\dot{\varphi}\sin\varphi$

⑤ $L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}_1^2 + 2\dot{x}_1\ell\dot{\varphi}\cos\varphi + \ell^2\dot{\varphi}^2) + m_2 g\ell\cos\varphi$

⑥ Lagr. rce. $(s=2)$

x_1: $0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \underbrace{\frac{\partial L}{\partial x_1}}_{=0} = \frac{d}{dt}\left(\underbrace{m_1\dot{x}_1 + m_2\dot{x}_1 + m_2\ell\dot{\varphi}\cos\varphi}_{p_{x_1} \text{ je I.P.}}\right) - 0 = \underline{(m_1+m_2)\ddot{x}_1 + m_2\ell(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) = 0}$ (1)

φ: $0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = \frac{d}{dt}(m_2\dot{x}_1\ell\cos\varphi + m_2\ell^2\dot{\varphi}) - (-m_2\dot{x}_1\ell\dot{\varphi}\sin\varphi - m_2 g\ell\sin\varphi) = m_2\ddot{x}_1\ell\cos\varphi - \cancel{m_2\dot{x}_1\ell\dot{\varphi}\sin\varphi} + m_2\ell^2\ddot{\varphi} + \cancel{m_2\dot{x}_1\ell\dot{\varphi}\sin\varphi} + m_2 g\ell\sin\varphi = 0$

$\underline{\ddot{x}_1\cos\varphi + \ell\ddot{\varphi} + g\sin\varphi = 0}$ (2)

Pohyb po elipse?

$p_{x_1} = (m_1+m_2)\dot{x}_1 + m_2\ell\dot{\varphi}\cos\varphi = C_1$ konst $\quad / \int dt$

$\int(m_1+m_2)\underbrace{\dot{x}_1 dt}_{dx_1} + m_2\ell\int\cos\varphi\underbrace{\dot{\varphi}dt}_{d\varphi} = \int C_1 dt = C_1 t + C_2$

$\int(m_1+m_2)dx_1 + m_2\ell\int\cos\varphi d\varphi = C_1 t + C_2$

$(m_1+m_2)x_1 + m_2\ell\sin\varphi = C_1 t + C_2$ $\quad$ $x_1 = x_2 - \ell\sin\varphi$ $\quad (y_2 = \ell\cos\varphi)$

$(m_1+m_2)x_2 - m_1\ell\sin\varphi = C_1 t + C_2$

$-m_1\ell\sin\varphi = (C_1 t + C_2) - (m_1+m_2)x_2 \quad /^2$

$m_1^2\ell^2(1-\cos^2\varphi) = m_1^2\ell^2(\sin^2\varphi) = ((C_1 t + C_2) - (m_1+m_2)x_2)^2$

$m_1^2\ell^2 - m_1^2 y_2^2 = ((C_1 t + C_2) - (m_1+m_2)x_2)^2$

$$\boxed{1 = \frac{y_2^2}{\ell^2} + \frac{((C_1 t + C_2) - (m_1+m_2)x_2)^2}{m_1^2\ell^2}}$$

C_1, C_2 z bč. pod. $(C_1 = 0)$

Perioda (2) ← (1) $\quad \ddot{x}_1 = \frac{-m_2\ell(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi)}{(m_1+m_2)}$

$\ell\ddot{\varphi} + \frac{m_2\ell(\dot{\varphi}^2\sin\varphi - \ddot{\varphi}\cos\varphi)}{m_1+m_2}\cos\varphi + g\sin\varphi = 0$ $\quad$ / Linearizace

$\sim\cancel{\dot{\varphi}^2\varphi}$

pro malé φ Taylor $\sin\varphi \sim \varphi$ $\quad \cos\varphi \sim 1$

$$\boxed{\ddot{\varphi}\left(1 - \frac{m_2}{m_1+m_2}\right) + \frac{g}{\ell}\varphi = 0}$$

$\frac{m_1}{m_1+m_2}$

$\ddot{\varphi} + \underbrace{\frac{g}{\ell}\frac{(m_1+m_2)}{m_1}}_{\omega^2}\varphi = 0$

$\varphi = \underline{A}\cos(\omega t + \underline{B})$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}\frac{m_1}{(m_1+m_2)}}$$