5

(1) $y' = f(x)$ $y(x) = ?$ $/ \int dx$

$$\int 1\,dy = \int \underbrace{y'\,dx}_{\frac{dy}{dx}dx = dy} = \int f(x)\,dx$$

$$\boxed{y + c = \int f(x)\,dx}$$

(2) $y' = f(y)$

$$\frac{1}{f(y)}\,y' = 1 \quad / \int dx$$

$$\int \frac{dy}{f(y)} = \int \frac{1}{f(y)}\underbrace{y'\,dx}_{dy} = \int 1\,dx = x + c$$

3) $y'' = f(x)$ $/ \int dx$

$$y' + c = \int 1\,dy' = \int \underbrace{y''\,dx}_{\frac{dy'}{dx}dx = dy'} = \int f(x)\,dx \to (1)$$

4) $y'' = f(y')$

$$\frac{1}{f(y')}\,y'' = 1 \quad / \int dx$$

$$\int \frac{dy'}{f(y')} = \int \frac{1}{f(y')}\underbrace{y''\,dx}_{dy'} = \int 1\,dx = x + c \to (1)$$

5) $y'' = f(y)$

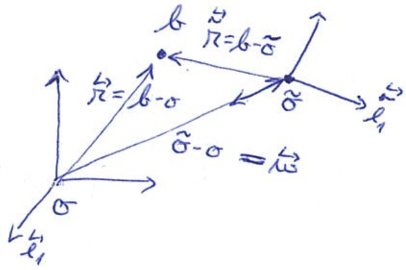
$$y'' = \frac{dy'}{dx} \underset{y' = y'(y(x))}{=} \frac{dy'}{dy}\cdot\frac{dy}{dx} = \frac{dy'}{dy}\cdot y'$$

$$\frac{dy'}{dy}\,y' = f(y) \quad / \int dy$$

$$\frac{1}{2}y'^2 + c = \int y'\,dy' = \int y' \underbrace{\frac{dy'}{dy}\,dy}_{dy'} = \int f(y)\,dy \to (2)$$

7 invertujte $x^j(b) = S^j_i(\tilde{x}^i(b) - \tilde{x}^i(o))$

↑ j-tá souřadnice bodu b v $\langle o, (\underbrace{\vec{e}_1, \dots, \vec{e}_n}_{B \,\dots\, \text{báze}})\rangle$ $\xrightarrow{S}$ $\langle \tilde{o}, (\tilde{\vec{e}}_1, \dots, \tilde{\vec{e}}_n)\rangle$ $\tilde{B} \dots$ báze



$$\tilde{\vec{e}}_j = \vec{e}_i S^i_j \qquad \text{"}\tilde{B} = B \cdot S\text{"}$$

1) $b = \tilde{o}$ $x^j(\tilde{o}) = S^j_i(\underbrace{\tilde{x}^i(\tilde{o})}_{0} - \tilde{x}^i(o)) = -S^j_i\tilde{x}^i(o)$

2) $x^j(b) = S^j_i\tilde{x}^i(b) - S^j_i\tilde{x}^i(o) = S^j_i\tilde{x}^i(b) + x^j(\tilde{o})$

$$x^j(b) - x^j(\tilde{o}) = S^j_i\tilde{x}^i(b) \quad / (S^{-1})^k_j$$

$$\to \boxed{(S^{-1})^k_j(x^j(b) - x^j(\tilde{o}))} = \underbrace{(S^{-1})^k_j S^j_i}_{(S^{-1}S)^k_i = \delta^k_i}\tilde{x}^i(b) = \delta^k_i\tilde{x}^i(b) = \boxed{\tilde{x}^k(b)}$$

Veličiny charakterizované podle chování při transformacích $S \in GL(n)$

1) skaláry $s \in \mathbb{R}$ $\tilde{s} = s$ $B = (\vec{e}_1, \dots, \vec{e}_n) \to \tilde{B} = (\tilde{\vec{e}}_1, \dots, \tilde{\vec{e}}_n)$

2) vektory $\vec{v} \in \vec{E} \to \vec{v} \in \mathbb{R}^n$ $\boxed{\tilde{v}^i = (S^{-1})^i_j v^j}$ $\tilde{\vec{v}} = S^{-1}\vec{v}$ $\boxed{\tilde{\vec{e}}_j = \vec{e}_i S^i_j}$ "$\tilde{B} = B \cdot S$"

kontravariantní $(\;) = (\vdots)(\;)$

kovektory $\underline{\alpha} \in \vec{E}^*$ (↑ duál k $\vec{E}$) $\to \vec{\alpha} \in \mathbb{R}^n$ $\boxed{\tilde{\alpha}_j = \alpha_i S^i_j}$ $\tilde{\vec{\alpha}}^T = \vec{\alpha}^T S$

kovariantní $(\dots) = (\dots)(\vdots)$

3, Tenzory

$\tilde{v}, \tilde{u} \in \tilde{E}$ v lib. bázi

Př: Tenzor 2. řádu $T^{ij} := v^i u^j$

$$\tilde{T}^{ij} = \tilde{v}^i \tilde{u}^j = (S^{-1})^i{}_k v^k (S^{-1})^j{}_l u^l = (S^{-1})^i{}_k (S^{-1})^j{}_l \overbrace{v^k u^l}^{T^{kl}} = (S^{-1})^i{}_k (S^{-1})^j{}_l T^{kl}$$

pomocí matic $(\tilde{T})^{ij} = (S^{-1}T)^{il}(S^{-1})^j{}_l = \left(\left((S^{-1}T)^T\right)^{li} \underbrace{(S^{-1})^j{}_l}\right) = \left(\underbrace{S^{-1}(S^{-1}T)^T}_{\text{transpozice}}\right)^{ji} = \left(S^{-1}T(S^{-1})^T\right)^{ij}$

$$\boxed{\tilde{T} = S^{-1}T(S^{-1})^T}$$ kontravariantní tenzor 2. řádu

Tenzor typu $\binom{p}{q}$ — p kontravariantní, q kovariantní

řád tenzoru $p+q$

$$\tilde{T}^{i_1,\dots,i_p}{}_{j_1,\dots,j_q} = \underbrace{(S^{-1})^{i_1}{}_{k_1} \cdots (S^{-1})^{i_p}{}_{k_p} T^{k_1,\dots,k_p}{}_{l_1,\dots,l_q} S^{l_1}{}_{j_1} \cdots S^{l_q}{}_{j_q}}_{\sum_{k_1,k_2,\dots,k_p,\ l_1,\dots,l_q}}$$

*4, Tenzorové hustoty

typu $\binom{p}{q}$ váhy $\lambda \in \mathbb{Z}$, $p,q \in \mathbb{N}_0$

$$\tilde{\tau}^{i_1,\dots,i_p}{}_{j_1,\dots,j_q} = (\det S)^\lambda (S^{-1})^{i_1}{}_{k_1} \cdots (S^{-1})^{i_p}{}_{k_p} \tau^{k_1,\dots,k_p}{}_{l_1,\dots,l_q} S^{l_1}{}_{j_1} \cdots S^{l_q}{}_{j_q}$$

8

$\tilde{e}_1 = e_1 + e_2 + e_3$

$\tilde{e}_2 = e_1 - e_2 + e_3$

$\tilde{e}_3 = e_1 - e_3$

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$\vec{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\tilde{\vec{A}} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$\vec{B} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ $\tilde{\vec{B}} = \begin{pmatrix} 15 \\ 5 \\ -2 \end{pmatrix}$

(e_1, e_2, e_3) $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$

Vektor $\tilde{\vec{x}} = S^{-1}\vec{x} \Leftrightarrow S\tilde{\vec{x}} = \vec{x}$

Kovektor $\tilde{\vec{x}}^T = \vec{x}^T S$

$\tilde{e}_j = e_k S^k{}_j$

① $\vec{A} = S\tilde{\vec{A}}$? $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ✓ může být vektor

② $\vec{B} = S\tilde{\vec{B}}$? $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 15 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 18 \\ 10 \\ 22 \end{pmatrix}$ × není vektor

$\tilde{\vec{B}}^T = \vec{B}^T S$.

$(4\ 3\ 6)\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} = (15\ 5\ -2)$ ✓ může být kovektor

9. D.C.V.

10. $(e_1, \dots, e_n)$ je OrtoNormální báze, jak vypadá skalární součin v lib. bázi $f_i = e_j F^j{}_i$, $F \in GL(n)$

$$\vec{a}\cdot\vec{b} = (a^i f_i)\cdot(b^j f_j) = a^i b^j (f_i \cdot f_j) = a^i b^j \left((e_k F^k{}_i)\cdot(e_l F^l{}_j)\right) =$$

$$= a^i b^j F^k{}_i F^l{}_j \underbrace{(e_k \cdot e_l)}_{\delta_{kl}} = a^i b^j F^k{}_i F^l{}_j \delta_{kl} = a^i b^j \underbrace{F^k{}_i F^k{}_j} = g_{ij} a^i b^j$$

g_{ij} ... prvky Gramovy matice $(f_1, \dots, f_n)$, $g_{ij} = f_i \cdot f_j$

11. Metrický tenzor $\tilde{f}_j = f_k S^i{}_j$

$$\boxed{\tilde{g}_{ij} = \tilde{f}_i \cdot \tilde{f}_j = (f_k S^k{}_i)\cdot(f_l S^l{}_j) = \underbrace{(f_k \cdot f_l)}_{= g_{kl}} S^k{}_i S^l{}_j = \boxed{g_{kl} S^k{}_i S^l{}_j}}$$

tenzor typu $\binom{0}{2}$ tj. 2× kovariantní tenzor

$(g_{ij}) = \mathbb{g}$ $\quad (\tilde{\mathbb{g}})_{ij} = (\mathbb{g})_{kl}\, S^k{}_i\, S^l{}_j = S^k{}_i\, (\mathbb{g} S)_{kj} = \left((\mathbb{g} S)^T\right)_{jk} S^k{}_i = \left((\mathbb{g} S)^T S\right)_{ji} = (S^T \mathbb{g} S)_{ij}$

$$\boxed{\tilde{\mathbb{g}} = S^T \mathbb{g}\, S}$$

14. $g := \det(\mathbb{g}) = \det((g_{ij}))$ jaká je to veličina???

$$\tilde{g} = \det(\tilde{\mathbb{g}}) = \det(S^T \mathbb{g} S) = \underbrace{\det S^T}_{=\det S} \det \mathbb{g} \det S = (\det S)^{2} \det \mathbb{g} = (\det S)^2 g$$

tenzorová hustota řádu 0 (skalární) váhy $\lambda = 2$

15. def. v lib. bázi $T^i{}_j := \delta^i_j \;\begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ je $GL(n)$-invariantní tenzor typu $\binom{1}{1}$

①

$$\delta^i_j = \tilde{T}^i{}_j = (S^{-1})^i{}_k\, T^k{}_l\, S^l{}_j = (S^{-1})^i{}_k\, \delta^k_l\, S^l{}_j = (S^{-1})^i{}_k\, S^k{}_j = (\underbrace{S^{-1} S}_{\mathbb{1}})^i{}_j = \delta^i_j \;\checkmark$$

② def. $T_{ij} := \delta_{ij}$ ← pouze v O.N. bázích

$$\delta_{ij} = \tilde{T}_{ij} = T_{kl}\, S^k{}_i\, S^l{}_j = \delta_{kl}\, S^k{}_i\, S^l{}_j = S^k{}_i\, S^k{}_j = \delta_{ij}$$

pouze pro $S \in O(n)$ je $O(n)$-invariantní tenzor typu $\binom{0}{2}$

③ def. $T_{ijk} := \varepsilon_{ijk}$

$$\varepsilon_{ijk} = \tilde{T}_{ijk} = (\det S)^{-1} T_{lmn}\, S^l{}_i\, S^m{}_j\, S^n{}_k = (\det S)^{-1} \underbrace{\varepsilon_{lmn}\, S^l{}_1\, S^m{}_2\, S^n{}_3}_{\varepsilon_{ijk} \det S} = \cancel{(\det S)^{-1}}\, \varepsilon_{ijk} \cancel{\det S} = \varepsilon_{ijk} \;\checkmark$$

a) $SO(n)$-invariantní tenzor řádu $\binom{0}{3}$ def. byla pouze v O.N. bázích ($\det S = 1$)

b) $GL(n)$-invariantní tenzorová hustota typu $\binom{0}{3}$ váhy $\lambda = -1$

DCV. $T^{ijk} := \varepsilon^{ijk}$

13. $U = U(\vec{x})$ skalární pole $\rightarrow$ $\nabla U(\vec{x})$ se transformují jako vektorové pole při $\boxed{\text{indexy dole}}$ ← ortogonálních transformacích tj. $S \in O(n)$

v O.N. bázi

$\boxed{\tilde{U}(\tilde{\vec{x}}) = U(\vec{x})}$

$$\boxed{\left(\tilde{\nabla} \tilde{U}(\tilde{\vec{x}})\right)_i} = \frac{\partial \tilde{U}(\tilde{\vec{x}})}{\partial \tilde{x}_i} = \frac{\partial U(\vec{x})}{\partial \tilde{x}_i} = \frac{\partial U(\vec{x})}{\partial x_j} \frac{\partial x_j}{\partial \tilde{x}_i} = \frac{\partial U(\vec{x})}{\partial x_j} \frac{\partial \left(S_{jl}(\tilde{x}_l - \tilde{x}_l(0))\right)}{\partial \tilde{x}_i} =$$

Řetízkové pravidlo

$x^j = S^j{}_l (\tilde{x}^l - \tilde{x}^l(0)) \rightarrow \boxed{x_j = S_{jl}(\tilde{x}_l - \tilde{x}_l(0))}$

$$= \frac{\partial U(\vec{x})}{\partial x_j} S_{jl} \Big[\underbrace{\frac{\partial \tilde{x}_l}{\partial \tilde{x}_i}}_{\delta_{li}} - \underbrace{\frac{\partial \tilde{x}_l(0)}{\partial \tilde{x}_i}}_{0} \Big] = \frac{\partial U(\vec{x})}{\partial x_j} S_{jl}\, \delta_{li} = \frac{\partial U(\vec{x})}{\partial x_j} S_{ji} = \underline{\left(\nabla U(\vec{x})\right)_j S_{ji}} \;\checkmark$$

pro $S \in O(n)$ $\quad S^T = S^{-1} \quad \tilde{\vec{v}} = S^{-1} \vec{v} = S^T \vec{v} \iff \tilde{\vec{v}}^T = \vec{v}^T S$

při O.G. transformacích se kovariantní a kontravariantní veličiny transformují stejně.