Lagrangeovy rovnice 1. druhu $m_i\ddot{x}_i = F_i(\vec{x},\dot{\vec{x}},t) + \sum_{k=1}^{r}\lambda_k \frac{\partial f_k}{\partial x_i} + \underbrace{\sum_{k=1}^{r} T_i^{(k)}}_{=0}$ $\quad \forall i\in\widehat{3N}$ $\qquad f_k(\vec{x},t)=0 \quad \forall k\in\hat{r}$

1. Operátor úplné časové derivace $\frac{\hat{d}}{dt}: C^\infty(\mathbb{R}^{k\cdot s+1}) \longrightarrow C^\infty(\mathbb{R}^{(k+1)\cdot s+1})$ $\qquad \frac{\hat{d}}{dt}: F(\vec{x},\dot{\vec{x}},\dots,\vec{x}^{(k)},t) \longrightarrow \dot{F}(\vec{x},\dot{\vec{x}},\dots,\vec{x}^{(k+1)},t)$

pro $F:\mathbb{R}^{2s+1}\to\mathbb{R}$ $\quad F=F(\vec{x},\dot{\vec{x}},t)$

$\vec{x},\dot{\vec{x}},t$ nezávislé proměnné

$$\dot{F} = \frac{\hat{d}F}{dt} = \frac{\partial F}{\partial x_i}\dot{x}_i + \frac{\partial F}{\partial \dot{x}_i}\ddot{x}_i + \frac{\partial F}{\partial t}\Big|_{(\vec{x},\dot{\vec{x}},t)}$$

$\dot{F} = \dot{F}(\underbrace{\vec{x},\dot{\vec{x}},\ddot{\vec{x}},t}_{})$ nezávislé proměnné $\dot{\vec{x}}=\vec{v}$, $\ddot{\vec{x}}=\vec{a}$

2. parciální derivace podle času $\quad \frac{\partial F}{\partial t} = \lim_{\Delta t\to 0}\frac{F(\vec{x},\dot{\vec{x}},t+\Delta t)-F(\vec{x},\dot{\vec{x}},t)}{\Delta t}$

3. úplná derivace podle času

$\vec{x}=\vec{X}(t)$, $\dot{\vec{x}}=\dot{\vec{X}}(t)$ $\quad \tilde{F}(t)=F(\vec{X}(t),\dot{\vec{X}}(t),t)$ ← funkce času

$$\dot{\tilde{F}} = \frac{d\tilde{F}}{dt} = \frac{\partial F}{\partial x_i}\dot{x}_i(t) + \frac{\partial F}{\partial \dot{x}_i}\ddot{x}_i(t) + \frac{\partial F}{\partial t}\Big|_{(\vec{X}(t),\dot{\vec{X}}(t),t)}$$

$$\begin{tikzcd}
F(\vec{x},\dot{\vec{x}},t) \arrow[r, "\frac{\hat{d}}{dt}"] \arrow[d, "\vec{x}=\vec{X}(t)"] & \dot{F}(\vec{x},\dot{\vec{x}},\ddot{\vec{x}},t) \arrow[d, "\vec{x}=\vec{X}(t)"] \\
\underbrace{F(\vec{X}(t),\dot{\vec{X}}(t),t)}_{\tilde{F}(t)} \arrow[r, "\frac{d}{dt}"'] & \underbrace{\dot{F}(\vec{X}(t),\dot{\vec{X}}(t),\ddot{\vec{X}}(t),t)}_{\dot{\tilde{F}}(t)}
\end{tikzcd}$$

- upravíme levou stranu LR1D $\quad \forall i\in\widehat{3N}$ $\quad$ (tj. bez sumace přes i)

$$m_i\ddot{x}_i = m_i\frac{\hat{d}}{dt}(\dot{x}_i) = \frac{\hat{d}}{dt}(m_i\dot{x}_i) = \frac{\hat{d}}{dt}\left(\tfrac{1}{2}m_i 2\dot{x}_i\right) = \frac{\hat{d}}{dt}\left(\sum_j \tfrac{1}{2}m_j 2\dot{x}_j\delta_{ji}\right) = \frac{\hat{d}}{dt}\left(\sum_j \tfrac{1}{2}m_j 2\dot{x}_j\frac{\partial \dot{x}_j}{\partial \dot{x}_i}\right) = \frac{\hat{d}}{dt}\left(\frac{\partial}{\partial \dot{x}_i}\underbrace{\left(\tfrac{1}{2}\sum_j m_j\dot{x}_j^2\right)}_{T(\dot{\vec{x}})}\right)$$

$$= \frac{\hat{d}}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)$$

kde $T(\dot{\vec{x}}) = \frac{1}{2}\sum_{j=1}^{3N} m_j\dot{x}_j^2$ je kinetická energie soustavy

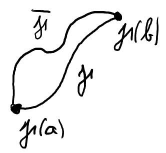
- Síly vtištěné (akční) na pravé staně LR1D nahradíme potenciály → silové pole (síla) $\vec{F}$ se nazývá:

1) <u>konzervativní</u> $\vec{F}=\vec{F}(\vec{x})$ pokud $\exists U=U(\vec{x})$ potenciální energie tak, že $F_j(\vec{x}) = -\frac{\partial U(\vec{x})}{\partial x_j}$ $\quad \forall j\in\hat{s}$

2) <u>potenciální</u> $\vec{F}=\vec{F}(\vec{x},t)$ pokud $\exists U=U(\vec{x},t)$ potenciál tak, že $F_j(\vec{x},t) = -\frac{\partial U(\vec{x},t)}{\partial x_j}$ $\quad \forall j\in\hat{s}$

3) <u>zobecněná potenciální</u> $\vec{F}=\vec{F}(\vec{x},\dot{\vec{x}},t)$ pokud $\exists U=U(\vec{x},\dot{\vec{x}},t)$ tak, že $F_j(\vec{x},\dot{\vec{x}},t) = -\frac{\partial U}{\partial x_j} + \frac{\hat{d}}{dt}\left(\frac{\partial U}{\partial \dot{x}_j}\right)$ $\quad \forall j\in\hat{s}$
zobecněný potenciál

Pozn. práce konzervativních sil nezávisí na dráze $\quad W = \int_\gamma \vec{F}(\vec{x})\cdot d\vec{x} = -\int_\gamma \frac{\partial U}{\partial \vec{x}}d\vec{x} = -\int_\gamma 1\,dU = -\int_a^b 1\,dU(\gamma(\ell)) = U(\gamma(a)) - U(\gamma(b))$



Pozn. v $\mathbb{R}^3$ podmínka $\mathrm{rot}\,\vec{F}=0 \Rightarrow \vec{F}(\vec{x},t)$ je potenciální

Př. – homogenní tíhové pole $\quad U(\vec{x}) = -m\vec{g}\cdot\vec{x}$

– centrální gravitační pole $\quad U(\vec{x}) = -\varkappa\frac{Mm}{|\vec{x}|}$

– harmonický ocilátor (elastické pole) $\quad U(\vec{x}) = \frac{1}{2}K\left(\sqrt{\vec{x}^2}-a_0\right)^2$

– Lorentzova síla (E.M. pole)

$U(\vec{x},\dot{\vec{x}},t) = q\left(\varphi(\vec{x},t) - \dot{\vec{x}}\cdot\vec{A}(\vec{x},t)\right)$

$\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$ $\qquad \vec{B} = \nabla\times\vec{A}$

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

- LR1D $\quad \frac{\hat{d}}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = F_i^{(nep)} + \frac{\hat{d}}{dt}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i} + \lambda_k\frac{\partial f_k}{\partial x_i}$ $\qquad T=T(\dot{\vec{x}})$ tj. $\frac{\partial T}{\partial x_i}=0$

$$\frac{\hat{d}}{dt}\left(\frac{\partial(T-U)}{\partial \dot{x}_i}\right) - \frac{\partial(T-U)}{\partial x_i} = F_i^{(nep)} + \lambda_k\frac{\partial f_k}{\partial x_i}$$

$\forall i\in\widehat{3N}$ $\quad \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = F_i^{(nep)} + \lambda_k\frac{\partial f_k}{\partial x_i}$ $\qquad f_k(\vec{x},t)=0 \quad \forall k\in\hat{r}$

Lagrangeova funkce (v kartézských)

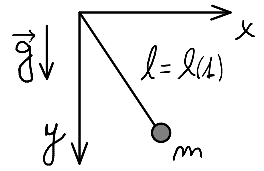
$$\boxed{\begin{gathered} L = T - U \\ L(\vec{x},\dot{\vec{x}},t) = \frac{1}{2}\sum_{j=1}^{3N} m_j\dot{x}_j^2 - U(\vec{x},\dot{\vec{x}},t)\end{gathered}}$$

• nejednoznačnost Lagrangeovy funkce $L' = L + \frac{\hat{d}}{dt} h(\vec{x},t)$ $\quad h = h(\vec{x},t) \in C^{(2)}$ $\quad \frac{\hat{d}h}{dt} = \frac{\partial h}{\partial x_j}\dot{x}_j + \frac{\partial h}{\partial t}$

$$\frac{\hat d}{dt}\left(\frac{\partial L'}{\partial \dot x_i}\right) - \frac{\partial L'}{\partial x_i} = \underbrace{\frac{\hat d}{dt}\left(\frac{\partial L}{\partial \dot x_i}\right) - \frac{\partial L}{\partial x_i}} + \frac{\hat d}{dt}\left(\frac{\partial h}{\partial \dot x_i}\right) - \frac{\partial}{\partial x_i}\frac{\hat d h}{dt} = \qquad \frac{\partial}{\partial \dot x_i}\left(\frac{\hat d h}{dt}\right) = \frac{\partial h}{\partial x_j}\frac{\partial \dot x_j}{\partial \dot x_i} = \frac{\partial h}{\partial x_i}$$

$$= LS + \frac{\partial^2 h}{\partial x_j \partial x_i}\dot x_j + \frac{\partial^2 h}{\partial t\,\partial x_i} - \frac{\partial^2 h}{\partial x_i\partial x_j}\dot x_j - \frac{\partial^2 h}{\partial x_i \partial t} = LS \Rightarrow$$ L a L' vedou na stejné LR1D

Př.



$T = \frac12 m(\dot x^2 + \dot y^2)$

$U = -m\vec g\cdot\vec x = -mgy$

$L = T - U = \frac12 m(\dot x^2+\dot y^2) + mgy$

$\frac{\hat d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = \frac{\hat d}{dt}(m\dot x) - 0 = m\ddot x = 2\lambda x$

$\frac{\hat d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) - \frac{\partial L}{\partial y} = \frac{\hat d}{dt}(m\dot y) - mg = m\ddot y - mg = 2\lambda y$

$f(x,y,t) = x^2+y^2 - l^2(t) = 0$

Křivočaré souřadnice v afinním euklidovském prostoru E dimenze $n\in\mathbb N$

$\vec X = \hat{\vec X}(\vec y, t)$ $\quad X^i = \hat X^i(y^1,\dots,y^n,t)$ $\quad \forall i\in\hat n$ $\quad \hat X^i: \mathbb R^{n+1}\to\mathbb R^n$ třídy $C^{(2)}$, regulární

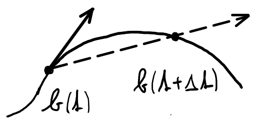
kartézské složky polohového vektoru

souřadnice, n-tice čísel nejsou složky vektoru

Jacobián $J = \det\left(\frac{\partial \hat X^i}{\partial y^j}\right) = \left|\frac{\partial(\hat X^1,\dots,\hat X^n)}{\partial(y^1,\dots,y^n)}\right| \neq 0$

tečný vektor

$\lim_{\Delta t\to 0}\frac{b(t+\Delta t) - b(t)}{\Delta t}$



Báze $B = \mathbb X = (\overleftarrow{e}_1,\dots,\overleftarrow{e}_n)$ a $\tilde B = \mathbb Y = (\overleftrightarrow{e}_1,\dots,\overleftrightarrow{e}_n)$ tvořená tečnými vektory k souřadnicovým křivkám $y^j \to \hat{\vec X}(\vec y, t)$

$\overleftrightarrow{e}_j = \overleftarrow{e}_i S^i{}_j \quad (\overleftrightarrow{e}_j)_B = \frac{\partial \vec X}{\partial y^j} \Rightarrow \overleftrightarrow{e}_j = \tilde\partial_j = \frac{\partial}{\partial y^j} = \frac{\partial \hat X^i}{\partial y^j}\frac{\partial}{\partial x^i} = \frac{\partial\hat X^i}{\partial y^j}\partial_i = S^i{}_j \overleftarrow{e}_i \qquad \mathbb S = \left(\frac{\partial \hat X^i}{\partial y^j}\right) = \left(\frac{\partial\vec X}{\partial y^1},\dots,\frac{\partial \vec X}{\partial y^n}\right)$

$\overleftarrow{e}_j = \overleftrightarrow{e}_i (S^{-1})^i{}_j \quad (\overleftarrow{e}_j)_{\tilde B} = \frac{\partial\vec y}{\partial x^j} \Rightarrow \overleftarrow e_j = \partial_j = \frac{\partial}{\partial x^j} = \frac{\partial \hat y^i}{\partial x^j}\frac{\partial}{\partial y^i} = \frac{\partial \hat y^i}{\partial x^j}\tilde\partial_i = (S^{-1})^i{}_j \overleftrightarrow{e}_i \qquad \mathbb S^{-1} = \left(\frac{\partial\hat y^i}{\partial x^j}\right) = \left(\frac{\partial \vec y}{\partial x^1},\dots,\frac{\partial\vec y}{\partial x^n}\right)$

Př. polární souřadnice (nenormalizované)

$x_1 = y_1\cos y_2 \quad y_1\in(0,+\infty)$

$x_2 = y_1 \sin y_2 \quad y_2\in\langle 0,2\pi)$

$\mathbb S = \left(\frac{\partial x_i}{\partial y_j}\right) = \begin{pmatrix}\cos y_2 & -y_1\sin y_2\\ \sin y_2 & y_1\cos y_2\end{pmatrix} \qquad y_1 = \sqrt{x_1^2+x_2^2} \quad y_2 = \mathrm{arctg}\left(\frac{x_2}{x_1}\right) \qquad \mathbb S^{-1} = \left(\frac{\partial y_i}{\partial x_j}\right) = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2}\end{pmatrix}$

Newtonovy rovnice v křivočarých souřadnicích (pro jednu částici s hmotností m v prostoru $\mathbb R^n$)

$m\ddot x^i = F^i(\vec x,\dot{\vec x},t) \qquad x^i = \hat x^i(\vec y,t) \qquad \dot x^i = \hat{\dot x}^i(\vec y,\dot{\vec y},t) = \frac{\hat d}{dt}\hat x^i(\vec y,t) = \frac{\partial\hat x^i}{\partial y^j}\dot y^j + \frac{\partial \hat x^i}{\partial t} \qquad \forall i\in\hat n$

$$\ddot x^i = \hat{\ddot x}^i(\vec y,\dot{\vec y},\ddot{\vec y},t) = \frac{\hat d}{dt}\left(\frac{\partial \hat x^i}{\partial y^j}\dot y^j + \frac{\partial\hat x^i}{\partial t}\right) = \frac{\partial^2\hat x^i}{\partial y^k\partial y^j}\dot y^j\dot y^k + \frac{\partial \hat x^i}{\partial y^j}\frac{\partial \dot y^j}{\partial \dot y^k}\ddot y^k + \frac{\partial^2 \hat x^i}{\partial t\,\partial y^j}\dot y^j + \frac{\partial^2\hat x^i}{\partial y^k\partial t}\dot y^k + \frac{\partial^2\hat x^i}{\partial t^2} =$$

$$= \frac{\partial^2\hat x^i}{\partial y^k\partial y^j}\dot y^j\dot y^k + \frac{\partial\hat x^i}{\partial y^j}\ddot y^j + 2\frac{\partial^2\hat x^i}{\partial t\,\partial y^j}\dot y^j + \frac{\partial^2 \hat x^i}{\partial t^2}$$

$y^l = \hat y^l(\vec x,t)$ inverzní transformace

$$m\left[\frac{\partial^2\hat x^i}{\partial y^k\partial y^j}\dot y^j\dot y^k + \frac{\partial\hat x^i}{\partial y^j}\ddot y^j + 2\frac{\partial^2\hat x^i}{\partial t\,\partial y^j}\dot y^j + \frac{\partial^2 \hat x^i}{\partial t^2}\right] = F^i(\vec x,\dot{\vec x},t) \quad \Big/ (S^{-1})^l{}_i = \frac{\partial \hat y^l}{\partial x^i} \qquad \frac{\partial\hat y^l}{\partial x^i}\frac{\partial\hat x^i}{\partial y^j} = \frac{\partial y^l}{\partial y^j} = \delta^l_j$$

$$m\frac{\partial\hat y^l}{\partial x^i}\underbrace{\left[\frac{\partial^2\hat x^i}{\partial y^k\partial y^j}\dot y^j\dot y^k + 2\frac{\partial^2\hat x^i}{\partial t\,\partial y^j}\dot y^j + \frac{\partial^2\hat x^i}{\partial t^2}\right]}_{\text{polynom 2. stupně v rychlostech } \overset{?}{=} 0} + m\ddot y^l = \frac{\partial\hat y^l}{\partial x^i}F^i(\vec x,\dot{\vec x},t) = \tilde F^l(\vec y,\dot{\vec y},t) \qquad \forall l\in\hat n$$

$\Rightarrow$ Galileiho transformace je lineární funkcí souřadnic a času

Př. polární souřadnice (nenormalizované)

$\frac{\partial x_i}{\partial t} = 0 \qquad \left(\frac{\partial^2 x_1}{\partial y_k\partial y_j}\right) = \begin{pmatrix}0 & -\sin y_2\\ -\sin y_2 & -y_1\cos y_2\end{pmatrix} \qquad \left(\frac{\partial^2 x_2}{\partial y_k\partial y_j}\right) = \begin{pmatrix}0&\cos y_2\\ \cos y_2 & -y_1\sin y_2\end{pmatrix} \qquad \left(\frac{\partial y_l}{\partial x_i}\right) = \mathbb S^{-1} = \begin{pmatrix}\cos y_2 & \sin y_2\\ -\frac{\sin y_2}{y_1} & \frac{\cos y_2}{y_1}\end{pmatrix}$

$$m\begin{pmatrix}\cos y_2 & \sin y_2\\ -\frac{\sin y_2}{y_1} & \frac{\cos y_2}{y_1}\end{pmatrix}\begin{bmatrix}-2\dot y_1\dot y_2\sin y_2 - y_1\dot y_2^2\cos y_2\\ 2\dot y_1\dot y_2\cos y_2 - y_1\dot y_2^2\sin y_2\end{bmatrix} + m\begin{pmatrix}\ddot y_1\\ \ddot y_2\end{pmatrix} = \begin{pmatrix}F_1\cos y_2 + F_2\sin y_2\\ -\frac{F_1}{y_1}\sin y_2 + \frac{F_2}{y_1}\cos y_2\end{pmatrix} \Rightarrow \begin{aligned} m(\ddot y_1 - y_1\dot y_2^2) &= \tilde F_1\\ m\left(\ddot y_2 + \frac{2}{y_1}\dot y_1\dot y_2\right) &= \tilde F_2\end{aligned}$$