Hamiltonova funkce a Hamiltonovy rovnice

Lagr. fce $L = L(\vec{q}, \dot{\vec{q}}, t)$

Lagr. rce $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = 0$ $\quad \forall i \in \hat{s}$ $\Bigg\} \to$

O.D.R. 2. řádu

Ham. form. 1) obecná hybnost $\hat{p}_j = \frac{\partial L}{\partial \dot q_j}(\vec q, \dot{\vec q}, t) \to \dot q_j = \hat{\dot q}_j(\vec q, \vec p, t)$

2) Hamiltonián $H(\vec q, \vec p, t) = \vec p \cdot \hat{\dot{\vec q}} - \hat L = \hat E(\vec q, \vec p, t)$

$\hat L(\vec q, \vec p, t) = L(\vec q, \hat{\dot{\vec q}}(\vec q, \vec p, t), t)$

3) Ham. rce.

I. $\forall i \in \hat s \quad \dot q_i = \frac{\partial H}{\partial p_i} \quad \left(\Leftrightarrow p_j = \frac{\partial L}{\partial \dot q_j}\right)$

II. $\dot p_i = -\frac{\partial H}{\partial q_i} \quad (\Leftrightarrow$ Newton Pohyb. rce.$)$

4) Odvoďte Hamiltonovy rovnice přímo výpočtem derivací Hamiltonovy funkce.

$$\frac{\partial H}{\partial p_i} = \frac{\partial}{\partial p_i}\left(p_j \hat{\dot q}_j - \hat L\right) = \delta_{ij}\hat{\dot q}_j + \cancel{p_j \frac{\partial \hat{\dot q}_j}{\partial p_i}} - \underbrace{\cancel{\frac{\partial L}{\partial \dot q_j}}}_{p_j}\cancel{\frac{\partial \hat{\dot q}_j}{\partial p_i}} = \hat{\dot q}_i$$

$$\frac{\partial H}{\partial q_i} = \frac{\partial}{\partial q_i}\left(p_j \hat{\dot q}_j - \hat L\right) = \underbrace{\frac{\partial p_j}{\partial q_i}}_{=0}\hat{\dot q}_j + p_j \frac{\partial \hat{\dot q}_j}{\partial q_i} - \frac{\partial \hat L}{\partial q_i} = \cancel{p_j \frac{\partial \hat{\dot q}_j}{\partial q_i}} - \left(\frac{\partial L}{\partial q_i} + \underbrace{\cancel{\frac{\partial L}{\partial \dot q_j}}}_{p_j}\cancel{\frac{\partial \hat{\dot q}_j}{\partial q_i}}\right) = -\frac{\partial L}{\partial q_i}\bigg|_{\dot{\vec q} = \hat{\dot{\vec q}}} \overset{\text{Lagr. Rce}}{=} -\frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot q_i}\right)}_{p_i} = -\frac{d}{dt}p_i = -\dot p_i$$

5) Sestavte Hamiltonovy rovnice pro pohyb volného hmotného bodu v poli konzervativních sil a ukažte, že získané rovnice jsou ekvivalentní s rovnicemi Newtonovými.

$$L = T - U = \frac{1}{2} m \dot x_i^2 - U(\vec x) \qquad p_j = \frac{\partial L}{\partial \dot q_j} = \frac{1}{2} m \dot x_i 2 \delta_{ij} = m \dot x_j \to \dot x_j = \frac{p_j}{m}$$

$$H = p_j \dot x_j - L = E(\vec x, \dot{\vec x}, t) \quad \text{Dosadit} \qquad E = \frac{1}{2} m \dot x_i^2 + U(\vec x) = \frac{1}{2}\left(\frac{p_i}{m}\right)^2 m + U(\vec x)$$

$$\boxed{H = \frac{p_i^2}{2m} + U(\vec x)} \quad / \ \vec q \equiv \vec x$$

I. $\dot x_j = \frac{\partial H}{\partial p_j} = \frac{2 p_i \delta_{ij}}{2m} = \frac{p_j}{m}$ $\quad \left(\begin{matrix} x_i = x_i(t) \\ p_i = p_i(t) \end{matrix} ?\right)$

II. $\dot p_j = -\frac{\partial H}{\partial x_j} = -\frac{\partial U}{\partial x_j}$ $\quad \Leftrightarrow$ Newton. rce. $\ddot x_j = \frac{F_j}{m}$ $\quad (x_j = x_j(t)\ ?)$

$p_j = m \dot x_j \qquad \dot p_j = m \ddot x_j = -\frac{\partial U}{\partial x_j} = F_j$

6) Napište Hamiltonovu funkci harmonického oscilátoru (LHO)

$$L = \frac{1}{2} m \dot x^2 - \frac{1}{2} k x^2 \qquad p = \frac{\partial L}{\partial \dot x} = m \dot x \Rightarrow \dot x = \frac{p}{m} \qquad E = \frac{\partial L}{\partial \dot x}\dot x - L \overset{\text{Zase}}{=} T + U = \frac{1}{2} m \dot x^2 + \frac{1}{2} k x^2$$

$$\boxed{H = \frac{1}{2m} p^2 + \frac{1}{2} k x^2}$$

I. $\dot x = \frac{\partial H}{\partial p} = \frac{p}{m}$

II. $\dot p = -\frac{\partial H}{\partial x} = -\frac{1}{2} k 2x = -kx$

7) Napište Hamiltonovu funkci a sestavte Hamiltonovy rovnice částice s nábojem e a hmotností m v daném vnějším elektromagnetickém poli. $L = \frac{1}{2} m \dot x_i^2 - e(\varphi - \dot x_i A_i) \quad \varphi = \varphi(\vec x, t) \quad \vec A = \vec A(\vec x, t)$

$$p_j = \frac{\partial L}{\partial \dot x_j} = m \dot x_j - e(-\delta_{ij} A_i) = m \dot x_j + e A_j \Rightarrow \dot x_j = \frac{p_j - e A_j}{m}$$

$$E = \frac{\partial L}{\partial \dot x_j}\dot x_j - L = (m \dot x_j + e A_j)\dot x_j - \left(\frac{1}{2} m \dot x_i^2 - e(\varphi - \dot x_i A_i)\right) = m \dot x_j^2 + \cancel{e A_j \dot x_j} - \frac{1}{2} m \dot x_i^2 + e\varphi - \cancel{e \dot x_i A_i} = \frac{1}{2} m \dot x_j^2 + e\varphi$$

$$H = \frac{1}{2m}(p_i - e A_i)^2 + e\varphi$$

I. $\dot x_j = \frac{\partial H}{\partial p_j} = \frac{1 \cdot 2}{2m}(p_i - e A_i)\delta_{ij} = \frac{p_j - e A_j}{m}$

II. $\dot p_j = -\frac{\partial H}{\partial x_j} = -\left[\frac{1}{2m} 2 (p_i - e A_i)\left(-e \frac{\partial A_i}{\partial x_j}\right) + e \frac{\partial \varphi}{\partial x_j}\right] = +\frac{(p_i - e A_i)}{m} e \frac{\partial A_i}{\partial x_j} - e \frac{\partial \varphi}{\partial x_j}$

8) Napište Hamiltonovu funkci volného hmotného bodu v a) kartézských b) sférických c) cylindrických souřadnicích. v poli $U(\vec{x})$

a) $L = \frac{1}{2} m \dot{x}_i^2 - U(\vec{x}) \rightarrow H = \frac{1}{2m} p_i^2 + U(\vec{x})$

b) $x_1 = r \sin\theta \cos\varphi$, $x_2 = r \sin\theta \sin\varphi$, $x_3 = r\cos\theta$

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2 \sin^2\theta + r^2\dot{\theta}^2\right) - U(r,\theta,\varphi)$$

$$E = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\varphi}^2 \sin^2\theta + r^2\dot{\theta}^2\right) + U(r,\theta,\varphi)$$

$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \dot{r} = \frac{p_r}{m}$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \rightarrow \dot{\theta} = \frac{p_\theta}{m r^2}$

$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \sin^2\theta\, \dot{\varphi} \rightarrow \dot{\varphi} = \frac{p_\varphi}{m r^2 \sin^2\theta}$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2\sin^2\theta} + \frac{p_\theta^2}{r^2}\right) + U(r,\theta,\varphi)$$

$m\begin{pmatrix}1 & & \\ & R^2 & \\ & & 1\end{pmatrix}$

c) $x_1 = R\cos\varphi$, $x_2 = R\sin\varphi$, $x_3 = z$

$\dot{x}_1 = \dot{R}\cos\varphi - R\dot{\varphi}\sin\varphi$, $\dot{x}_2 = \dot{R}\sin\varphi + R\dot{\varphi}\cos\varphi$, $\dot{x}_3 = \dot{z}$

$$L = \frac{1}{2} m\left(\dot{R}^2 + R^2\dot{\varphi}^2 + \dot{z}^2\right) - U(R,\varphi,z)$$

$$E = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \frac{1}{2} m\left(\dot{R}^2 + R^2\dot{\varphi}^2 + \dot{z}^2\right) + U(R,\varphi,z)$$

$p_R = \frac{\partial L}{\partial \dot{R}} = m\dot{R} \rightarrow \dot{R} = \frac{p_R}{m}$

$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mR^2\dot{\varphi} \rightarrow \dot{\varphi} = \frac{p_\varphi}{mR^2}$

$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \rightarrow \dot{z} = \frac{p_z}{m}$

$$H = \frac{1}{2m}\left(p_R^2 + R^2\left(\frac{p_\varphi}{R^2}\right)^2 + p_z^2\right) + U = \frac{1}{2m}\left(p_R^2 + \frac{p_\varphi^2}{R^2} + p_z^2\right) + U(R,\varphi,z)$$

$\frac{1}{m}\begin{pmatrix}1 & & \\ & \frac{1}{R^2} & \\ & & 1\end{pmatrix}$

9) Sestavte Hamiltonovy rovnice pro pohyb volného hmotného bodu pod vlivem centrální síly s potenciálem $U(r)$ ve sférických souřadnicích. Určete integrály pohybu.

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + U(r)$$

I. $r \leftrightarrow p_r$

$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$, $\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m r^2 \sin^2\theta}$

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m r^2}$

$0 = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \Rightarrow E$ je I.P.

$\Downarrow$

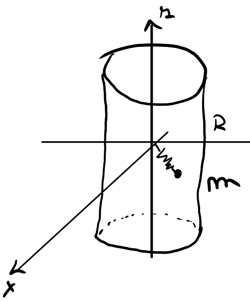
H je IP

II. $\dot{p}_r = -\frac{\partial H}{\partial r} = -\left[\frac{1}{2m}\left(\frac{p_\theta^2}{r^3}(-2) + \frac{(-2)p_\varphi^2}{r^3\sin^2\theta}\right) + \frac{\partial U}{\partial r}\right] = \frac{p_\theta^2}{m r^3} + \frac{p_\varphi^2}{m r^3 \sin^2\theta} - \frac{\partial U}{\partial r}$

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\left[\frac{1}{2m}\frac{p_\varphi^2}{r^2}(-2)\frac{1}{\sin^3\theta}\cos\theta\right] = \frac{p_\varphi^2 \cos\theta}{m r^2 \sin^3\theta}$

$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \rightarrow \boxed{p_\varphi = \text{konst.}}$ je I.P.

10) Hmotný bod m je vázán na válcovou plochu $x^2+y^2=R^2$ a pohybuje se po ní pod vlivem centrální elastické síly $\vec{F} = -k\vec{r}$. Najděte Hamiltonovu funkci, sestavte a řešte Hamiltonovy rovnice a řešte je (v cylindrických souřadnicích).



$U = \frac{1}{2}k(x^2+y^2+z^2)$

$$L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2}k(x^2+y^2+z^2)$$

obecné s.: $s = 3 - 1 = 2$

$x = \tilde{R}\cos\varphi$, $y = \tilde{R}\sin\varphi$, $z = z$

$x^2+y^2=R^2$

$\tilde{R}^2\cos^2\varphi + \tilde{R}^2\sin^2\varphi = R^2$

$\tilde{R}^2 = R^2 \Rightarrow \tilde{R} = R = \text{konst.}$

$\rightarrow$ obecné $\boxed{\varphi, z}$

$x = R\cos\varphi$, $\dot{x} = -R\dot{\varphi}\sin\varphi$

$y = R\sin\varphi$, $\dot{y} = R\dot{\varphi}\cos\varphi$

$z = z$, $\dot{z} = \dot{z}$

$$L = \frac{1}{2} m(R^2\dot{\varphi}^2 + \dot{z}^2) - \frac{1}{2}k(R^2+z^2)$$

obecné hybnosti

$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mR^2\dot{\varphi}$

$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$

$\dot{\varphi} = \frac{p_\varphi}{mR^2}$

$\dot{z} = \frac{p_z}{m}$

I. $\dot{q}_i = \frac{\partial H}{\partial p_i}$

$$E = \frac{\partial L}{\partial \dot{\varphi}}\dot{\varphi} + \frac{\partial L}{\partial \dot{z}}\dot{z} - L = \frac{1}{2} m(R^2\dot{\varphi}^2 + \dot{z}^2) + \frac{1}{2}k(R^2+z^2)$$

$$H = \frac{1}{2m}\left(\frac{p_\varphi^2}{R^2} + p_z^2\right) + \frac{1}{2}k(R^2+z^2)$$

II. $\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{1}{2}k\,2z = -kz$

$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \Rightarrow p_\varphi = \text{konst.}$ I.P.

$\int dt$ $\boxed{\varphi = \frac{p_\varphi}{mR^2}t + \varphi_0}$

$m\ddot{z} = -kz$

$\ddot{z} + \frac{k}{m}z = 0 \Rightarrow \boxed{z(t) = A\cos\left(\sqrt{\frac{k}{m}}t + B\right)}$