Hamiltonův princip pro pole: Skutečný časový vývoj soustavy polí se děje s takovou závislostí q_α na x^μ, pro kterou akce S nabývá stacionární hodnoty vzhledem k variacím $\delta q_\alpha(x^\mu)$ splňujícím podmínku pevných konců, která požaduje nulovost variací na hranici ∂V^* oblasti V^* tj: $\delta q_\alpha(x^\mu)\big|_{\partial V^*} = 0$.

kde akce

$$S(q_\alpha(x^\mu)) = \frac{1}{c}\int_{V^*} \underset{\uparrow \text{ hustota Lagrangeovy funkce}}{\mathcal{L}(q_\alpha, q_{\alpha,\mu}, x^\mu)}\, dV^* \qquad q_{\alpha,\mu} = \frac{\partial q_\alpha}{\partial x^\mu} = \partial_\mu q_\alpha \qquad dV^* = dx^0 dx^1 dx^2 dx^3 = c\,dt\,dV$$

je-li $V^* = \langle ct_1, ct_2\rangle \times V$ tak $S = \int_{V^*} \mathcal{L}\, dV^* = \int_{t_1}^{t_2}\int_V \mathcal{L}\, dV dt$ ($\int_V \mathcal{L}\,dV = L$; $\mathcal{L}$ … hustota Lagrangiánu)

$$\delta S = \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_\alpha}\delta q_\alpha + \frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\delta q_{\alpha,\mu}\right] dV^* = \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_\alpha}\delta q_\alpha + \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\delta q_\alpha\right) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\right)\delta q_\alpha\right] dV^* =$$

$$= \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_\alpha} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\right]\delta q_\alpha dV^* + \underbrace{\frac{1}{c}\int_{V^*}\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\delta q_\alpha\right) dV^*}_{\text{čtyřdivergence}} = \frac{1}{c}\int_{V^*}\left[\frac{\partial \mathcal{L}}{\partial q_\alpha} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\right]\delta q_\alpha dV^* + \underbrace{\frac{1}{c}\int_{\partial V^*}\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\delta q_\alpha\, df_\mu}_{=0\ (\delta q_\alpha|_{\partial V^*}=0)}$$

df_μ … kovariantní složky orientovaného elementu 3-míry uzavřené nadplochy ∂V^* která je hranicí objemu V^*

Lagrangeovy rovnice $\left(\sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial q_{\alpha,\mu}}\right)\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \quad \forall \alpha = 1,2,\dots$

7.33 Odvoďte pohybové rovnice polí ve dvourozměrném prostoročase (t,x) ze zadaných hustot Lagrangiánu.

$$\frac{\partial}{\partial x^0} = \frac{\partial}{\partial(ct)} = \frac{\partial}{\partial t}\cdot\frac{\partial t}{\partial(ct)} = \frac{\partial \frac{1}{c}(ct)}{\partial(ct)}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial}{\partial t} \qquad \frac{\partial \mathcal{L}}{\partial q_{,0}} = \frac{\partial \mathcal{L}}{\partial(\frac{1}{c}q_{,t})} = c\frac{\partial \mathcal{L}}{\partial q_{,t}} \qquad \frac{\partial}{\partial x^0}\frac{\partial \mathcal{L}}{\partial q_{,0}} = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial q_{,t}}$$

$$q_{,0} = \frac{\partial q}{\partial x^0} = \frac{1}{c}\frac{\partial q}{\partial t} = \frac{1}{c}q_{,t}$$

místo q_α máme $\varphi = \varphi(t,x)$, $\varphi_t = \frac{\partial \varphi}{\partial t}$

1) $(\varphi^4)_2$: $\quad \mathcal{L} = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + \frac{1}{2}\mu^2\varphi^2 - \frac{1}{4}\lambda\varphi^4 \quad (\lambda, \mu^2 > 0)$

$$\frac{\partial \mathcal{L}}{\partial \varphi_t} = \varphi_t \qquad \frac{\partial \mathcal{L}}{\partial \varphi_x} = -\varphi_x \qquad \frac{\partial \mathcal{L}}{\partial \varphi} = \mu^2\varphi - \lambda\varphi^3$$

$$0 = \frac{\partial}{\partial t}\varphi_t + \frac{\partial}{\partial x}(-\varphi_x) - (\mu^2\varphi - \lambda\varphi^3) = \underline{\varphi_{tt} - \varphi_{xx} - \mu^2\varphi + \lambda\varphi^3 = 0}$$

2) sinus-Gordon $\quad \mathcal{L} = \frac{1}{2}(\varphi_t^2 - \varphi_x^2) + (\cos\varphi - 1)$

- rovnice vyšla při hledání povrchů s konstantní křivostí $K = -1$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \varphi_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \varphi_x}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \qquad \frac{\partial}{\partial t}(\varphi_t) + \frac{\partial}{\partial x}(-\varphi_x) + \sin\varphi = 0$$

$$\underline{\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0}$$

3) Born-Infeld $\quad \mathcal{L} = \sqrt{1 - \varphi_t^2 + \varphi_x^2}$

- nelineární elektrodynamika, cílem bylo odstranit divergenci vlastní energie elektronu v klasické elektrodynamice zavedením horní meze pro el. pole v prostoru.

$$\frac{\partial}{\partial t}\left(\frac{-2\varphi_t}{2\sqrt{1-\varphi_t^2+\varphi_x^2}}\right) + \frac{\partial}{\partial x}\left(\frac{2\varphi_x}{2\sqrt{1-\varphi_t^2+\varphi_x^2}}\right) - 0 = 0$$

$$-\frac{\varphi_{tt}\sqrt{} - \varphi_t\frac{-\varphi_t\varphi_{tt}+\varphi_x\varphi_{xt}}{\sqrt{}}}{1-\varphi_t^2+\varphi_x^2} + \frac{\varphi_{xx}\sqrt{} - \varphi_x\frac{-\varphi_t\varphi_{tx}+\varphi_x\varphi_{xx}}{\sqrt{}}}{1-\varphi_t^2+\varphi_x^2} = 0$$

$$-\varphi_{tt}(1-\varphi_t^2+\varphi_x^2) - \varphi_t^2\varphi_{tt} + \varphi_t\varphi_x\varphi_{xt} + \varphi_{xx}(1-\varphi_t^2+\varphi_x^2) + \varphi_x\varphi_t\varphi_{tx} - \varphi_x^2\varphi_{xx} = 0$$

$$\varphi_{xx} - \varphi_{tt} - \varphi_{xx}\varphi_t^2 - \varphi_{tt}\varphi_x^2 + 2\varphi_t\varphi_x\varphi_{tx} = 0$$

$$\varphi_{xx}(1-\varphi_t^2) - \varphi_{tt}(1+\varphi_x^2) + 2\varphi_t\varphi_x\varphi_{tx} = 0$$

4) Korteweg-de Vries

$$\mathcal{L} = \frac{1}{2}\theta_x\theta_t + \frac{\omega}{6}\theta_x^3 + \theta_x\psi_x + \frac{1}{2}\psi^2$$

zde jde o dvě pole $\psi = \psi(t,x)$, $\theta = \theta(t,x)$; budou dvě rovnice které dáme dohromady zavedením $\varphi = \theta_x$

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \psi_t} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \psi_x} - \frac{\partial \mathcal{L}}{\partial \psi} = \theta_{xx} - \psi = 0 \Rightarrow \psi = \theta_{xx}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \theta_t}\right) + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \theta_x} - \frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial}{\partial t}\left(\frac{1}{2}\theta_x\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}\theta_t + \frac{\omega}{2}\theta_x^2 + \psi_x\right) - 0 = \frac{1}{2}\theta_{xt} + \frac{1}{2}\theta_{tx} + \omega\theta_x\theta_{xx} + \psi_{xx} = 0$$

$$\theta_{xt} + \omega\theta_x\theta_{xx} + \underbrace{\psi_{xx}}_{\theta_{xxxx}} = 0$$

$$\underline{\varphi_t + \omega\varphi\varphi_x + \varphi_{xxx} = 0}$$

KdV – matematický model popisující vlny na mělké vodě

(7.32) Odvoďte pohybovou rovnici reálného skalárního pole $\varphi = \varphi(x^\mu)$ pokud $\mathcal{L} = \frac{1}{2}(\varphi_{,\mu}\varphi^{,\mu} - \mathcal{H}^2\varphi^2)$

$$\varphi_{,\mu} = \frac{\partial \varphi}{\partial x^\mu} \qquad \varphi^{,\mu} = \frac{\partial \varphi}{\partial x_\mu} = \frac{\partial \varphi}{\partial g_{\mu\nu}x^\nu} \overset{\text{díky } g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)}{=} \frac{1}{g_{\mu\nu}}\frac{\partial \varphi}{\partial x^\nu} = \underset{\uparrow \text{ inverse}}{g^{\mu\nu}}\frac{\partial \varphi}{\partial x^\nu} = g^{\mu\nu}\varphi_{,\nu} \qquad \text{tj. } \mathcal{L} = \frac{1}{2}(g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - \mathcal{H}^2\varphi^2)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi_{,\rho}} = \frac{1}{2}g^{\mu\nu}(\delta_\mu^{\ \rho}\varphi_{,\nu} + \varphi_{,\mu}\delta_\nu^{\ \rho}) = \frac{1}{2}g^{\rho\nu}\varphi_{,\nu} + \frac{1}{2}g^{\mu\rho}\varphi_{,\mu} = g^{\rho\nu}\varphi_{,\nu}$$

$$\frac{\partial}{\partial x^\rho}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,\rho}}\right) = \frac{\partial}{\partial x^\rho}(g^{\rho\nu}\varphi_{,\nu}) = g^{\rho\nu}\varphi_{,\nu\rho}$$

$$g^{\rho\nu}\varphi_{,\nu\rho} + \mathcal{H}^2\varphi = 0$$

$$\varphi_{,00} - \varphi_{,11} - \varphi_{,22} - \varphi_{,33} + \mathcal{H}^2\varphi = 0$$

$$\frac{1}{c^2}\varphi_{tt} - \Delta\varphi + \mathcal{H}^2\varphi = -\Box\varphi + \mathcal{H}^2\varphi = 0$$