30. Co říká věta o viriálu pro lineární harmonický oscilátor a pro Coulombické pole?

$\langle T\rangle = \frac{1}{2}k\langle U\rangle$

$U(\alpha\vec{x}) = \alpha^k U(\vec{x})$

$U = \frac{1}{2}kx^2$ homogenní st. 2

$\langle T\rangle = \frac{2}{2}\langle U\rangle = \langle U\rangle$

$U = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ st. $k=-1$

$r = \sqrt{\sum x_i^2}$

$\langle T\rangle = -\frac{1}{2}\langle U\rangle$

25. Spočtěte $\vec{\tilde{\Omega}} = ?$ je-li matice přechodu $S = \begin{pmatrix}\cos\varphi(t) & -\sin\varphi(t) & 0\\ \sin\varphi(t) & \cos\varphi(t) & 0\\ 0 & 0 & 1\end{pmatrix} \in SO(3)$

$\tilde{\omega} = -S^T\dot{S}$ $\quad \tilde{\Omega}_i = \frac{1}{2}\varepsilon_{ijk}\tilde{\omega}_{jk} = -\frac{1}{2}\varepsilon_{ijk}(S^T\dot{S})_{jk}$

$$\tilde{\omega} = -\begin{pmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\dot{\varphi}(-\sin\varphi) & -\dot{\varphi}\cos\varphi & 0\\ \dot{\varphi}\cos\varphi & -\dot{\varphi}\sin\varphi & 0\\ 0 & 0 & 0\end{pmatrix} = -\begin{pmatrix}\dot{\varphi}(\cancel{-\cos\varphi\sin\varphi+\sin\varphi\cos\varphi}) & -\dot{\varphi}(\cos^2\varphi+\sin^2\varphi) & 0\\ \dot{\varphi} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\tilde{\omega} = \dot{\varphi}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\tilde{\Omega}_1 = \frac{1}{2}\varepsilon_{1jk}\tilde{\omega}_{jk} = \frac{1}{2}(\varepsilon_{123}\tilde{\omega}_{23} + \underbrace{\overset{=-\varepsilon_{123}}{\varepsilon_{132}}\overset{=-\tilde{\omega}_{23}}{\tilde{\omega}_{32}}}_{(-1)(-\tilde{\omega}_{23})}) = \frac{1}{2}(\tilde{\omega}_{23}+\tilde{\omega}_{23}) = \tilde{\omega}_{23} = 0$$

$$\tilde{\Omega}_2 = \frac{1}{2}(\underset{=1}{\varepsilon_{231}}\tilde{\omega}_{31} + \underset{=-1}{\varepsilon_{213}}\tilde{\omega}_{13}) = \tilde{\omega}_{31} = 0$$

$$\tilde{\Omega}_3 = \frac{1}{2}(\underset{=1}{\varepsilon_{312}}\tilde{\omega}_{12} + \underset{=-1}{\varepsilon_{321}}\tilde{\omega}_{21}) = \tilde{\omega}_{12} = 1\dot{\varphi}$$

$$\vec{\tilde{\Omega}} = \begin{pmatrix}0\\0\\\dot{\varphi}\end{pmatrix}$$

26. Ukažte, že pro libovolné $\vec{\tilde{y}} \in \mathbb{R}^3$ platí $-\tilde{\omega}_{ij}\tilde{y}_j = (S^T\dot{S})_{ij}\tilde{y}_j = (\vec{\tilde{\Omega}}\times\vec{\tilde{y}})_i$ $\quad (\tilde{\omega}^2)_{ik}\tilde{y}_k = (\vec{\tilde{\Omega}}\times(\vec{\tilde{\Omega}}\times\vec{\tilde{y}}))_i$

$\underline{-\tilde{\omega}_{ij}}\tilde{y}_j = \underline{\varepsilon_{ilj}\tilde{\Omega}_l}\tilde{y}_j = (\vec{\tilde{\Omega}}\times\vec{\tilde{y}})_i$

$\tilde{\Omega}_i = \frac{1}{2}\varepsilon_{ijk}\tilde{\omega}_{jk} \quad /\varepsilon_{ilm}$

$\varepsilon_{ilm}\tilde{\Omega}_i = \frac{1}{2}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\tilde{\omega}_{jk}$

$= \frac{1}{2}(\tilde{\omega}_{lm} - \underset{\nabla(-1)}{\tilde{\omega}_{ml}}) = \frac{1}{2}2\tilde{\omega}_{lm}$

$(\tilde{\omega}\tilde{\omega})_{ik}\tilde{y}_k = \tilde{\omega}_{il}\tilde{\omega}_{lk}\tilde{y}_k$

$= \tilde{\omega}_{il}(-\vec{\tilde{\Omega}}\times\vec{\tilde{y}})_l =$

$= +(\vec{\tilde{\Omega}}\times(\vec{\tilde{\Omega}}\times\vec{\tilde{y}}))_i$

27. Ukažte, že pro $\vec{\tilde{e}}_i(t) = \vec{e}_j S_{ji}(t)$ $S(t)\in SO(3)$ platí $\dot{\vec{\tilde{e}}}_i = \tilde{\omega}_{ij}\vec{\tilde{e}}_j$ a jak se transformují složky $\vec{Y}(t)$?

$$\dot{\vec{\tilde{e}}}_i = \underset{=0}{\dot{\vec{e}}_j}S_{ji} + \vec{e}_j\dot{S}_{ji} = \vec{e}_j\dot{S}_{ji} = S_{jl}\vec{\tilde{e}}_l\dot{S}_{ji} = (S^T)_{lj}\dot{S}_{ji}\vec{\tilde{e}}_l = \underbrace{(S^T\dot{S})_{li}}_{-\tilde{\omega}_{li}}\vec{\tilde{e}}_l = -\underbrace{\tilde{\omega}_{li}}_{=-\tilde{\omega}_{il}}\vec{\tilde{e}}_l = +\tilde{\omega}_{il}\vec{\tilde{e}}_l$$

$S_{ki}\vec{\tilde{e}}_i = \vec{e}_j S_{ji}S_{ki} = \vec{e}_j\delta_{jk} = \vec{e}_k$

$-\tilde{\omega}_{lj}\tilde{Y}_j = (\vec{\tilde{\Omega}}\times\vec{\tilde{Y}})_l$

$$\vec{Y}(t) = Y_i(t)\vec{e}_i = \tilde{Y}_j(t)\vec{\tilde{e}}_j(t) \quad /\frac{d}{dt} \quad \Rightarrow \dot{\vec{Y}}(t) = \dot{Y}_i\vec{e}_i = \dot{\tilde{Y}}_j\vec{\tilde{e}}_j + \tilde{Y}_j\dot{\vec{\tilde{e}}}_j = \dot{\tilde{Y}}_l\vec{\tilde{e}}_l + \tilde{Y}_j\tilde{\omega}_{jl}\vec{\tilde{e}}_l = (\dot{\tilde{Y}}_l + \tilde{\omega}_{jl}\tilde{Y}_j)\vec{\tilde{e}}_l =$$

$$\dot{Y}_i S_{il}\vec{\tilde{e}}_l = \qquad = (\dot{\vec{\tilde{Y}}} + (\vec{\tilde{\Omega}}\times\vec{\tilde{Y}}))_l\vec{\tilde{e}}_l$$

$$\boxed{\dot{\vec{\tilde{Y}}} = S^T\dot{\vec{Y}} - \vec{\tilde{\Omega}}\times\vec{\tilde{Y}}}$$

maticově $\vec{\tilde{Y}} = S^T\vec{Y} \quad /\frac{d}{dt} \quad \dot{\vec{\tilde{Y}}} = \dot{S}^T\vec{Y} + S^T\dot{\vec{Y}} = \underbrace{\dot{S}^T S}_{=-S^T\dot{S}=\tilde{\omega}}\vec{\tilde{Y}} + S^T\dot{\vec{Y}} = \tilde{\omega}\vec{\tilde{Y}} + S^T\dot{\vec{Y}} = \underline{-\vec{\tilde{\Omega}}\times\vec{\tilde{Y}} + S^T\dot{\vec{Y}}}$

$S\vec{\tilde{Y}} = \underbrace{SS^T}_{\mathbb{1}}\vec{Y} = \vec{Y}$

28. Ukažte, že $\omega_{ij} = -(\dot{S}S^T)_{ij}$ a $\tilde{\omega}_{ij} = -(S^T\dot{S})_{ij}$ jsou složky téhož tenzoru v různých bázích a $\vec{\Omega}$ a $\vec{\tilde{\Omega}}$ složky téhož vektoru v různých bázích. $S\in SO(3)$

$\tilde{\omega}_{ij} = \omega_{kl}S_{ki}S_{lj}$ $\quad \tilde{\omega}_{ij} = S_{ki}(\omega S)_{kj} = (S^T)_{ik}(\omega S)_{kj} = (S^T\omega S)_{ij}$ $\quad \boxed{\tilde{\omega} = S^T\omega S}$

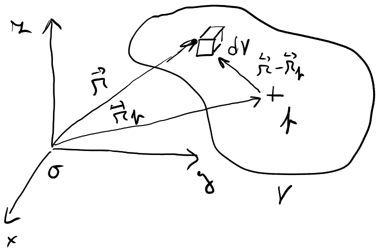
$\tilde{\omega} = S^T\omega S = S^T(-\underbrace{\dot{S}S^T)S}_{=\mathbb{1}} = -S^T\dot{S} = \tilde{\omega}$ ✓

$\tilde{\Omega}_i = -\frac{1}{2}\varepsilon_{ijk}(S^T\dot{S})_{jk}$ $\quad \Omega_i = -\frac{1}{2}\varepsilon_{ijk}(\dot{S}S^T)_{jk}$ $\quad (\dot{S}S^T)_{lr} = \dot{S}_{lm}S_{rm}$

$$\tilde{\Omega}_i = \Omega_j S_{ji} = -\frac{1}{2}\varepsilon_{jlm}(\dot{S}S^T)_{lm}S_{ji} = -\frac{1}{2}\underline{\varepsilon_{jlm}}\dot{S}_{lk}S_{mk}S_{ji} = -\frac{1}{2}\varepsilon_{jrm}\delta_{lr}S_{ji}S_{mk}\dot{S}_{lk} =$$

$$= -\frac{1}{2}\underbrace{\varepsilon_{jrm}\underset{1}{S_{ji}}\underset{2}{S_{rn}}\underset{3}{S_{mk}}}_{(\det S)\cdot\varepsilon_{ink}}\dot{S}_{lk}S_{ln} = -\frac{1}{2}\underset{S\in SO(3)\ =1}{(\det S)}\varepsilon_{ink}S_{ln}\dot{S}_{lk} = -\frac{1}{2}\varepsilon_{ink}(S^T\dot{S})_{nk} = \tilde{\Omega}_i$$ ✓

32. Ukažte, že pro těleso s konstantní hustotou hmoty je v homogenním silovém poli je moment sil vzhledem k hmotnému středu tělesa nulový.



$\downarrow \vec{g}$ $\quad d\vec{N}_{dV} = (\vec{r}-\vec{r}_A)\times d\vec{F}_{dV} = (\vec{r}-\vec{r}_A)\times\vec{g}\,\underbrace{\rho\, dV}_{dm_{dV}}$

homogenní pole

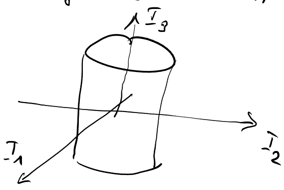
$$\vec{N}_A = \int_V (\vec{r}-\vec{r}_A)\times\vec{g}\,\rho\, dV = \left(\int_V (\vec{r}-\vec{r}_A)\rho\, dV\right)\times\vec{g} = \Big(\underbrace{\int\rho\,\vec{r}\, dV}_{M\vec{R}} - \vec{r}_A \underbrace{\int\rho\, dV}_{M\text{ . celk hmotnost}}\Big)\times\vec{g} =$$

$$= M(\vec{R}-\vec{r}_A)\times\vec{g} = \underline{0}$$

pro A = hm. střed $\vec{R} = \vec{r}_A$

33. Homogenní válec $(I_1 = I_2 \neq I_3)$ na který nepůsobí žádné síly rotuje s úhlovou rychlostí 10 rad/s okolo 2. osy. Jak bude rotovat za 10 s?

$$\vec{\Omega}(t_0) = \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix} \text{rad/s} \qquad \vec{\Omega}(t_0+10\,s) = ?$$



Eulerovy setrvačníkové rovnice

$$I_i \dot{\Omega}_i + \varepsilon_{ijk}\,\Omega_j I_k \Omega_k = N_i^{(e)} \quad \forall i = 1,2,3$$

bez sumace; v tělesové soustavě $(\sim)$

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$i = 1 \to 2 \to 3 \to 1$

$$I_1\dot{\Omega}_1 + (I_3 - I_2)\Omega_2\Omega_3 = N_1^{(e)}$$
$$I_2\dot{\Omega}_2 + (I_1 - I_3)\Omega_1\Omega_3 = N_2^{(e)}$$
$$I_3\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 = N_3^{(e)}$$

pro nás $\vec{N}^{(e)} = 0$

$I_1 = I_2 \neq I_3$

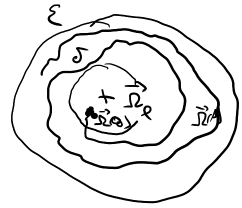
$$I_1\dot{\Omega}_1 + (I_3 - I_1)\Omega_2\Omega_3 = 0 \Rightarrow I_1\dot{\Omega}_1 = 0$$
$$I_1\dot{\Omega}_2 + (I_1 - I_3)\Omega_1\Omega_3 = 0 \Rightarrow I_2\dot{\Omega}_2 = 0$$
$$I_3\dot{\Omega}_3 + 0 = 0 \Rightarrow \dot{\Omega}_3 = 0$$

$$\Downarrow$$

$$0 = \Omega_3 = \text{konst}$$

$$\underline{\underline{\vec{\Omega}(t_0+10\,s) = \vec{\Omega}(t_0)}} \Leftarrow \begin{matrix}\Omega_1 = \text{konst} \\ \Omega_2 = \text{konst.}\end{matrix}$$

34. Ukažte, že rotace bezsilového setrvačníku je stabilní kolem os s maximálním a minimálním momentem setrvačnosti a nestabilní kolem osy se středním momentem setrvačnosti.



$\dot{\vec{\Omega}} = X(\vec{\Omega})$ ① Rovnovážný stav $\vec{\Omega}_e$ (def. $X(\vec{\Omega}_e) = 0$) je stabilní pokud
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall \vec{\Omega}(t)$ řešení ① platí $|\vec{\Omega}(0) - \vec{\Omega}_e| < \delta \Rightarrow |\vec{\Omega}(t) - \vec{\Omega}_e| < \varepsilon\ \forall t$

$\downarrow$ Linearizace $\vec{\Omega} = \vec{\Omega}_e + \delta\vec{\Omega}$

Taylor do 1. řádu kolem $\vec{\Omega}_e$

$$\frac{d(\vec{\Omega})}{dt} = \frac{d(\vec{\Omega}_e + \delta\vec{\Omega})}{dt} = \frac{d(\delta\vec{\Omega})}{dt} = X(\vec{\Omega}_e + \delta\vec{\Omega}) = \underbrace{X(\vec{\Omega}_e)}_{=0} + X'(\vec{\Omega}_e)\,\delta\vec{\Omega} + \cancel{\sigma^2(\delta\vec{\Omega})}$$

$$\frac{d(\delta\vec{\Omega})}{dt} = X'(\vec{\Omega}_e)\,\delta\vec{\Omega} \quad ②$$

Rovnovážný stav $\vec{\Omega}_e$ je lineárně stabilní pokud stav $\delta\vec{\Omega} = 0$ je stabilní pro systém ②.

Stabilita $\Rightarrow$ Lineární Stabilita (Lin. nestabilita $\Rightarrow$ nestabilita).