Kovariantní $\tilde{\alpha}_i = \alpha_j S^j{}_i$ pro $S \in O(n)$ $S^{-1} = S^T$

Kontravariantní $\tilde{v}^i = (S^{-1})^i{}_j v^j \overset{\downarrow}{=} (S^T)^i{}_j v^j = S^j{}_i v^j = v^j S^j{}_i$ $\Big\}$ pro $S \in O(n)$ stejné transformační vztahy

$\Downarrow$

indexy budou jen dole

Př: transformace $\mathrm{div}\,\vec F(\vec x)$ při O.G. Tr.

$$\breve{U}(\vec x) = \underbrace{\widetilde{\mathrm{div}}\,\tilde{\vec F}(\tilde{\vec x})} = \frac{\partial \tilde F_i(\tilde{\vec x})}{\partial \tilde x_i} \underset{\uparrow}{=} \frac{\partial (S_{ji} F_j(\vec x))}{\partial \tilde x_i} = S_{ji}\frac{\partial F_j(\vec x)}{\partial \tilde x_i} \underset{\uparrow}{=} S_{ji}\frac{\partial F_j(\vec x)}{\partial x_k}\frac{\partial x_k}{\partial \tilde x_i} \underset{\uparrow}{=} S_{ji}\frac{\partial F_j(\vec x)}{\partial x_k}\frac{\partial (S_{kl}(\tilde x_l - \tilde x_l(o)))}{\partial \tilde x_i} =$$

$\tilde{\vec F}(\tilde{\vec x}) = S^T \vec F(\vec x)$ Řetízkové pravidlo $\vec x = S(\tilde{\vec x} - \tilde{\vec x}(o))$

$x_k = S_{kl}(\tilde x_l - \tilde x_l(o))$

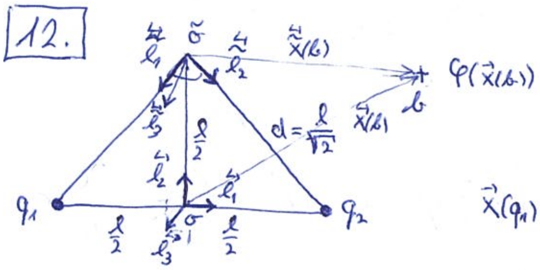
$$= S_{ji}\frac{\partial F_j(\vec x)}{\partial x_k} S_{kl}\Big(\underbrace{\frac{\partial \tilde x_l}{\partial \tilde x_i}}_{=\delta_{li}} - \underbrace{\frac{\partial \tilde x_l(o)}{\partial \tilde x_i}}_{=0}\Big) = S_{ji}\frac{\partial F_j(\vec x)}{\partial x_k} S_{kl}\delta_{li} \underset{\uparrow}{=} S_{ji} S_{ki}\frac{\partial F_j(\vec x)}{\partial x_k} = \delta_{jk}\frac{\partial F_j(\vec x)}{\partial x_k} =$$

$S \in O(n)$

$$= \frac{\partial F_j(\vec x)}{\partial x_j} = \underline{\mathrm{div}\,\vec F(\vec x)} = U(\vec x)$$

DCV. $\widetilde{\mathrm{rot}}\,\tilde{\vec F}(\tilde{\vec x}) = \ldots$

$\boxed{12.}$



Skalární elektrický potenciál $\left(\varphi(\vec r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec r|}\right)$ $\vec F = -\nabla\varphi$

$$\varphi_i(\vec x(b)) = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{\sqrt{(\vec x(b) - \vec x(q_i))^2}}$$

$$\vec x(q_1) = \begin{pmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{pmatrix} \quad \vec x(q_2) = \begin{pmatrix} \frac{l}{2} \\ 0 \\ 0 \end{pmatrix} \quad \tilde{\vec x}(q_1) = \begin{pmatrix} \frac{l}{\sqrt2} \\ 0 \\ 0 \end{pmatrix} \quad \tilde{\vec x}(q_2) = \begin{pmatrix} 0 \\ \frac{l}{\sqrt2} \\ 0 \end{pmatrix} \quad \vec x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\varphi(\vec x(b)) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\sqrt{(\vec x(b) - \vec x(q_1))^2}}\right) + \frac{1}{4\pi\varepsilon_0}\left(\frac{q_2}{\sqrt{(\vec x(b) - \vec x(q_2))^2}}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\underline{\sqrt{(x - (-\frac{l}{2}))^2 + y^2 + z^2}}} + \frac{q_2}{\sqrt{(x - \frac{l}{2})^2 + y^2 + z^2}}\right)$$

$$\tilde\varphi(\tilde{\vec x}(b)) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\sqrt{(\tilde{\vec x}(b) - \tilde{\vec x}(q_1))^2}} + \frac{q_2}{\sqrt{(\tilde{\vec x}(b) - \tilde{\vec x}(q_2))^2}}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\underline{\sqrt{(\tilde x - \frac{l}{\sqrt2})^2 + \tilde y^2 + \tilde z^2}}} + \frac{q_2}{\sqrt{(\tilde x)^2 + (\tilde y - \frac{l}{\sqrt2})^2 + \tilde z^2}}\right)$$

různé funkce popisují stejné pole

Transformace skalárního pole

$$\tilde\varphi(\tilde{\vec x}) = \varphi(\vec x) \quad \Rightarrow \quad \varphi(\underline{\vec x}) = \tilde\varphi(\tilde{\vec x}) = \tilde\varphi(\overbrace{S^T(\underline{\vec x} - \vec x(\tilde o))}^{\tilde{\vec x}}) \qquad \tilde{\vec l}_i = \vec l_j S_{ji} \qquad S = {}^{\tilde B}(Id)^B = \begin{array}{c} \begin{pmatrix} -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ -\frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \uparrow \quad \uparrow \quad \uparrow \\ (\tilde{\vec l}_1)_B (\tilde{\vec l}_2)_B (\tilde{\vec l}_3)_B \end{array}$$

$$\vec x(\tilde o) = \begin{pmatrix} 0 \\ \frac{l}{2} \\ 0 \end{pmatrix} \qquad \tilde{\vec x} = S^T(\vec x - \vec x(\tilde o)) = \begin{pmatrix} -\frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y - \frac{l}{2} \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt2}x - \frac{1}{\sqrt2}(y - \frac{l}{2}) \\ \frac{1}{\sqrt2}x - \frac{1}{\sqrt2}(y - \frac{l}{2}) \\ z \end{pmatrix}$$

$$\tilde\varphi(\ ) = \ldots \quad \Rightarrow \quad \underline{(\tilde x - \frac{l}{\sqrt2})^2 + \tilde y^2 + \tilde z^2} = \left(-\frac{x}{\sqrt2} - \frac{1}{\sqrt2}(y - \frac{l}{2}) - \frac{l}{\sqrt2}\right)^2 + \left(\frac{x}{\sqrt2} - \frac{1}{\sqrt2}(y - \frac{l}{2})\right)^2 + z^2 =$$

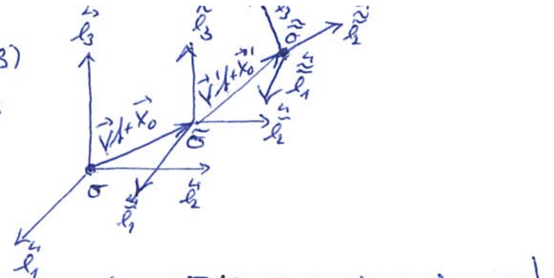
$$= \frac{x^2}{2} + \frac{1}{2}(y - \frac{l}{2})^2 + \frac{l^2}{2} + \cancel{x(y - \frac{l}{2})} + xl + l(y - \frac{l}{2}) + \frac{x^2}{2} - \cancel{x(y - \frac{l}{2})} + \frac{1}{2}(y - \frac{l}{2})^2 + z^2 =$$

$$= x^2 + (y - \frac{l}{2})^2 + xl + yl + z^2 = x^2 + y^2 - \cancel{yl} + \frac{l^2}{4} + xl + \cancel{yl} + z^2 = \underline{(x + \frac{l}{2})^2 + y^2 + z^2}$$

DCV. $\tilde x^2 + (\tilde y - \frac{l}{\sqrt2})^2 + \tilde z^2 = \ldots$

23. Galileiho transformace skládání

$$(\mathbb{S},\vec V,\vec x_0) \qquad \mathbb{S},\mathbb{S}'\in O(3)$$
$$(\mathbb{S}',\vec V',\vec x_0') \qquad \vec V,\vec V'\in\mathbb{R}^3 \qquad \vec x_0,\vec x_0'\in\mathbb{R}^3$$



$$\vec{\tilde x} = \mathbb{S}^T(\vec x - \vec x(\tilde\sigma)) = \mathbb{S}^T(\vec x - (\vec V t + \vec x_0)),$$

$$\vec{\tilde{\tilde x}} = \mathbb{S}'^T(\vec{\tilde x} - \vec{\tilde x}(\tilde{\tilde\sigma})) = \mathbb{S}'^T(\vec{\tilde x} - (\vec V' t + \vec x_0')) =$$

$$= \mathbb{S}'^T(\mathbb{S}^T[\vec x - \vec V t - \vec x_0] - (\vec V' t + \vec x_0')) = \mathbb{S}'^T\mathbb{S}^T(\vec x - \vec V t - \vec x_0 - \mathbb{S}\vec V' t - \mathbb{S}\vec x_0') = (\mathbb{S}\mathbb{S}')^T(\vec x - [(\vec V + \mathbb{S}\vec V')t + \vec x_0 + \mathbb{S}\vec x_0'])$$

$$\mathbb{I} = \mathbb{S}^T\mathbb{S}$$

$$\to \underline{(\mathbb{S}\mathbb{S}',\ \vec V + \mathbb{S}\vec V',\ \vec x_0 + \mathbb{S}\vec x_0')}$$

24. Transformace souřadnic $\vec r' = \vec r - \vec r(\sigma')$ (tj $\mathbb{S}=\mathbb{I}$) mezi soustavami, které mají // osy

$\vec r_\alpha' = \vec r_\alpha - \vec r(\sigma')$ ⇐ $x_j' = x_j - x_j(\sigma')$

a) pro moment hybnosti soustavy $N\in\mathbb{N}$ částic $\vec L = \sum_{\alpha=1}^N \vec l_\alpha = \sum_{\alpha=1}^N \vec r_\alpha\times\vec p_\alpha = \sum_{\alpha=1}^N \vec r_\alpha \times (m_\alpha \dot{\vec r}_\alpha)$

$$\vec L' = \sum_{\alpha=1}^N \vec l_\alpha' = \sum_\alpha \vec r_\alpha'\times\vec p_\alpha' = \sum_\alpha (\vec r_\alpha')\times(m_\alpha\dot{\vec r}_\alpha') = \sum_\alpha(\vec r_\alpha - \vec r(\sigma'))\times(\dot{\vec r}_\alpha - \dot{\vec r}(\sigma'))m_\alpha = \sum_\alpha \vec r_\alpha\times\dot{\vec r}_\alpha m_\alpha - \sum_\alpha \vec r_\alpha\times\dot{\vec r}(\sigma')m_\alpha$$

$$- \sum_\alpha \vec r(\sigma')\times\dot{\vec r}_\alpha m_\alpha + \sum_\alpha \vec r(\sigma')\times\dot{\vec r}(\sigma')m_\alpha = \underbrace{\sum_\alpha \vec r_\alpha\times(m_\alpha\dot{\vec r}_\alpha)}_{\vec L} - \underbrace{\left(\sum_\alpha m_\alpha\vec r_\alpha\right)}_{M\frac{\sum m_\alpha \vec r_\alpha}{\sum m_\alpha} = M\vec R}\times\dot{\vec r}(\sigma') - \vec r(\sigma')\times\underbrace{\sum_\alpha \underbrace{m_\alpha\dot{\vec r}_\alpha}_{\vec p_\alpha}}_{\vec P} + \underbrace{\left(\sum_\alpha m_\alpha\right)}_{M}\vec r(\sigma')\times\dot{\vec r}(\sigma')$$

$M\vec R$ – Hmotný střed; $\vec P$ celková hybnost; M hmotnost

$$\boxed{\vec L' = \vec L - M\vec R\times\dot{\vec r}(\sigma') - \vec r(\sigma')\times\vec P + M\vec r(\sigma')\times\dot{\vec r}(\sigma')}$$

b) dále nečárkovaná soustava je inerciální

specializace pro Galileiho transformaci $\vec r(\sigma') = \vec W t + \vec a$ $\quad\vec W,\vec a\in\mathbb{R}$ konst. $\quad\dot{\vec r}(\sigma') = \vec W$

$$\vec L' = \vec L - M\vec R\times\vec W - (\vec W t+\vec a)\times\vec P + M(\vec W t + \vec a)\times\vec W = \vec L - M\vec R\times\vec W - \vec W t\times\vec P - \vec a\times\vec P + M\vec a\times\vec W$$

Pro izolovanou soustavu platí $\dot{\vec L}=0$ $\dot{\vec P}=0$ ⇐ 1. a 2. věta impulsová v inerciální soustavě ⇒ $\dot{\vec L}'=0$

$$\dot{\vec L}' = \underbrace{\dot{\vec L}}_{0} - \underbrace{M\dot{\vec R}}_{\vec P}\times\vec W - M\vec R\times\underbrace{\dot{\vec W}}_{0} - \vec W\times\vec P - \vec W t\times\underbrace{\dot{\vec P}}_{0} - \underbrace{\dot{\vec a}}_{0}\times\vec P - \vec a\times\underbrace{\dot{\vec P}}_{0} + 0 = -\vec P\times\vec W - \vec W\times\vec P = \vec W\times\vec P - \vec W\times\vec P = 0 \ ✓$$

c) specializace pro soustavu hmotného středu $\vec r(\sigma') = \vec R$

$$\vec L' = \vec L - \cancel{M\vec R\times\dot{\vec R}} - \vec R\times\vec P + \cancel{M\vec R\times\dot{\vec R}} = \underline{\vec L - \vec R\times\vec P}$$

2. Věta impulsová → v soustavě hmotného středu $(\vec r_\alpha' = \vec r_\alpha - \vec R)$

v inerciální soustavě $\dot{\vec L} = \vec N^{(e)} = \sum_\alpha \vec r_\alpha\times\vec F_\alpha^{(e)}$ $\qquad \mathbb{S}=\mathbb{I} \Rightarrow \vec F'^{(e)}_\alpha = \vec F^{(e)}_\alpha$

$$\dot{\vec L}' + \underbrace{\dot{\vec R}\times\vec P}_{\dot{\vec R}\times M\dot{\vec R}=0} + \vec R\times\dot{\vec P} = (\vec L' + \vec R\times\vec P)^{\cdot} = \dot{\vec L} = \vec N^{(e)} = \sum_\alpha(\vec r_\alpha' + \vec R)\times\vec F'^{(e)}_\alpha = \underbrace{\sum_\alpha \vec r_\alpha'\times\vec F'^{(e)}_\alpha}_{\vec N'^{(e)}} + \vec R\times\underbrace{\sum_\alpha \vec F'^{(e)}_\alpha}_{\vec F^{(e)}} = \vec N'^{(e)} + \vec R\times\underbrace{\vec F^{(e)}}_{\dot{\vec P}}$$

⇐ 1. věta impulsová v inerciální soustavě

$$\dot{\vec L}' + \cancel{\vec R\times\dot{\vec P}} = \vec N'^{(e)} + \cancel{\vec R\times\dot{\vec P}}$$

$$\boxed{\dot{\vec L}' = \vec N'^{(e)}}$$ v soustavě hm. středu má stejný tvar jako v inerciální soustavě.

29. Věta o viriálu pro soustavu částic s náboji q a hmotnostmi m v homog. mag. poli o indukci $\vec{B}$, pohybujících se s omezenými rychlostmi v omezené části prostoru.

a v potenciálním poli U stupně k homogenním

(1) časová střední hodnota fce $f = f(t)$ $\quad \langle f \rangle = \lim\limits_{\tau \to +\infty} \frac{1}{\tau} \int_0^\tau f(t)\,dt$

pokud $\exists F,\ F' = f$, omezená $|F(x)| \leq K\ \forall x \Rightarrow \langle f \rangle = 0$

(2) Eulerova věta o homogenních funkcích $f(\vec{x})$ st. $k \in \mathbb{R}$ $\quad f(\lambda \vec{x}) = \lambda^k f(\vec{x}) \quad \forall \lambda > 0$

platí $\displaystyle \sum_{i=1}^{n} \frac{\partial f(\vec{x})}{\partial x_i} x_i = k f(\vec{x})$

(3) Viriálový teorém $\quad \left\langle \frac{dG}{dt} \right\rangle = 0 \quad \Rightarrow \quad \langle T \rangle = \underbrace{-\frac{1}{2} \left\langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle}_{\text{Viriál}}$

$G = \sum_\alpha \vec{p}_\alpha \cdot \vec{r}_\alpha$

Newtonovy rce $\quad \underset{\text{bez sumace}}{m_\alpha \ddot{\vec{r}}_\alpha} = \vec{F}_\alpha = q_\alpha (\underset{\overset{\parallel}{0}}{\vec{E}} + \vec{v}_\alpha \times \vec{B}) - \frac{\partial U}{\partial \vec{r}_\alpha} = q_\alpha \vec{v}_\alpha \times \vec{B} - \frac{\partial U}{\partial \vec{r}_\alpha}$

víme
$m_\alpha = m \quad \forall \alpha$
$q_\alpha = q \quad \forall \alpha$

$$\langle T \rangle = -\frac{1}{2} \left\langle \sum_\alpha \vec{F}_\alpha \cdot \vec{r}_\alpha \right\rangle = -\frac{1}{2} \left\langle \sum_\alpha \left(q \vec{v}_\alpha \times \vec{B} - \frac{\partial U}{\partial \vec{r}_\alpha} \right) \cdot \vec{r}_\alpha \right\rangle = -\frac{1}{2} \left\langle \sum \underbrace{q (\vec{v}_\alpha \times \vec{B}) \cdot \vec{r}_\alpha} \right\rangle - \frac{1}{2} \left\langle - \sum_\alpha \frac{\partial U}{\partial \vec{r}_\alpha} \cdot \vec{r}_\alpha \right\rangle =$$

$$\frac{m}{m} r_{\alpha i} \varepsilon_{ijk} v_{\alpha j} B_k = \frac{B_k \overbrace{\varepsilon_{kij} r_{\alpha i} p_{\alpha j}}^{l_{\alpha k}}}{m} = \frac{1}{m} \vec{B} \cdot \vec{l}_\alpha$$

$$\langle T \rangle = -\frac{1}{2} \left\langle \frac{q}{m} \Big(\underbrace{\sum_\alpha \vec{l}_\alpha}_{\vec{L}} \Big) \cdot \vec{B} \right\rangle + \frac{1}{2} \left\langle \underbrace{\sum_{\alpha=1}^{N} \sum_{i=1}^{3} \frac{\partial U}{\partial x_{\alpha i}} \cdot x_{\alpha i}}_{kU \text{ Euler. věta}} \right\rangle = \boxed{-\frac{1}{2} \frac{q}{m} \langle \vec{L} \rangle \cdot \vec{B} + \frac{1}{2} k \langle U \rangle}$$