# Hamiltonův formalizmus (pro holonomní soustavy a potenciální síly v obecných souřadnicích)

Pozn: Kanonický tvar obyčejných diferenciálních rovnic $y_i' = \frac{dy_i}{dx} = f_i(x, y_1, \dots, y_s) \quad \forall i \in \hat{s}$

LR2D $\frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} \underbrace{\dot{q}_j}_{v_j} + \underbrace{\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}}_{\mathbb{L}_{ij}} \underbrace{\ddot{q}_j}_{\dot{v}_j} + \frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad \forall i \in \hat{s} \longrightarrow$

$$\dot{q}_j = v_j \quad \forall j \in \hat{s}$$

$$\dot{v}_k = (\mathbb{L}^{-1})_{ki} \left[ \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} v_j \right]_{\dot{\vec{q}} = \vec{v}} \quad \forall k \in \hat{s}$$

$$\det \mathbb{L} = \left| \left( \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \right) \right| = \left| \left( \frac{\partial p_i}{\partial \dot{q}_j} \right) \right| \neq 0$$

Jinak

$$0 = \frac{d}{dt} \underbrace{\left( \frac{\partial L}{\partial \dot{q}_i} \right)}_{p_i} - \frac{\partial L}{\partial q_i} = \dot{p}_i - \frac{\partial L}{\partial q_i} \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}(\vec{q}, \dot{\vec{q}}, t) \longrightarrow \dot{q}_j = \hat{\dot{q}}_j(\vec{q}, \vec{p}, t) \quad \forall j \in \hat{s}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \bigg|_{(\vec{q}, \hat{\dot{\vec{q}}}(\vec{q}, \vec{p}, t), t)} \quad \forall i \in \hat{s}$$

$s$ ODR 2. řádu přechází na $2s$ ODR 1. řádu

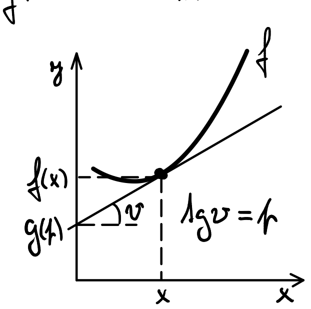
Přejdeme pomocí fce. L od nezávislých proměnných $(\vec{q}, \dot{\vec{q}}, t) \underset{H}{\overset{L}{\rightleftarrows}} (\vec{q}, \vec{p}, t)$ k novým nezávislým proměnným a najdeme funkci H, která zprostředkuje přechod zpět a pomocí ní zapíšeme rovnice.

## Legendreova duální transformace

$f: \mathbb{R} \to \mathbb{R} \qquad f = f(x) \qquad$ (konvexní) $f''(x) > 0 \quad \forall x \in \mathbb{R}$

$x \xrightarrow{f} p \qquad \hat{p} = \frac{df}{dx}(x) = f'(x)$ inverzní funkce $\qquad df = \frac{df}{dx} dx = \hat{p}\, dx$

$p \xrightarrow{g} \pm x$ (TEF, TSFa) $\qquad \pm \hat{x} = \frac{dg}{dp}(p) = g'(p) \qquad g'(f'(x)) = \pm x \qquad dg = \frac{dg}{dp} dp = \pm \hat{x}\, dp \quad \Big/ \pm$




$f(x) = px + g(p)$

$[x, f(x)]$

$\downarrow$

$[p, g(p)]$

$$d(f \pm g) = df \pm dg = p\,dx + x\,dp = d(px) \Rightarrow f \pm g = px + \underset{0}{\text{Konst.}}$$

$$g(p) = \pm (p \hat{x}(p) - \hat{f}(p)) \qquad \hat{f}(p) = f(\hat{x}(p))$$

$$f(x) = \hat{p}(x) x \mp \hat{g}(x) \qquad \hat{g}(x) = g(\hat{p}(x))$$

dualita transformace $x \overset{f}{\longrightarrow} p \overset{g}{\longrightarrow} x$

Legendreova transformace Lagrangeovy funkce $\oplus \qquad f \to L \quad x \to \dot{\vec{q}} \quad \frac{d}{dx} \to \frac{\partial}{\partial \dot{q}_i} \quad p \to \vec{p} \quad g \to H$

**1) Obecná hybnost** $\hat{p}_i = \frac{\partial L}{\partial \dot{q}_i}(\vec{q}, \dot{\vec{q}}, t) \Longrightarrow \dot{q}_i = \hat{\dot{q}}_i(\vec{q}, \vec{p}, t)$ lze pokud $\det \left( \frac{\partial \hat{p}_i}{\partial \dot{q}_j} \right) = \underbrace{\det \left( \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \right)}_{\text{hessián}} \neq 0$

**2) Hamiltonova funkce** $H(\vec{q}, \vec{p}, t) = \sum_{j=1}^{s} p_j \hat{\dot{q}}_j - \hat{L} = E(\vec{q}, \hat{\dot{\vec{q}}}(\vec{q}, \vec{p}, t), t)$ kde $\hat{L}(\vec{q}, \vec{p}, t) = L(\vec{q}, \hat{\dot{\vec{q}}}(\vec{q}, \vec{p}, t), t)$

obecná energie v proměnných $(\vec{q}, \vec{p}, t)$

**3) Hamiltonovy kanonické rovnice** (kanonický tvar LR2D)

I. sada $\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}}$ nemá dynamický obsah, je ekvivalentní definici obecné hybnosti

$\forall i \in \hat{s}$

$$\dot{q}_i = \hat{\dot{q}}_i(\vec{q}, \vec{p}, t) = \frac{\partial H}{\partial p_i} \qquad \hat{H}(\vec{q}, \dot{\vec{q}}, t) = H(\vec{q}, \hat{\vec{p}}(\vec{q}, \dot{\vec{q}}, t), t) = E(\vec{q}, \dot{\vec{q}}, t)$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i} = \frac{\partial}{\partial q_i}(\hat{p}_j \dot{q}_j - \hat{H}) = \frac{\partial \hat{p}_j}{\partial q_i} \dot{q}_j - \frac{\partial \hat{H}}{\partial q_i} = \frac{\partial \hat{p}_j}{\partial q_i} \underline{\dot{q}_j} - \frac{\partial H}{\partial q_i} - \underline{\frac{\partial H}{\partial p_j}} \frac{\partial \hat{p}_j}{\partial q_i} = -\frac{\partial H}{\partial q_i}$$

II. sada $\boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i}}$ má dynamický obsah, nahrazuje Newtonovy rovnice

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}(p_j \hat{\dot{q}}_j - \hat{L}) = \underline{p_j} \frac{\partial \hat{\dot{q}}_j}{\partial t} - \frac{\partial L}{\partial t} - \underline{\frac{\partial L}{\partial \dot{q}_j}} \frac{\partial \hat{\dot{q}}_j}{\partial t} = -\frac{\partial L}{\partial t} \qquad \boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \bigg|_{\dot{\vec{q}} = \hat{\dot{\vec{q}}}(\vec{q}, \vec{p}, t)}}$$

Př. Konzervativní soustava (konzervativní síly a skleronomní vazby): $L(\vec{q}, \dot{\vec{q}}, t) = \frac{1}{2} T_{jk}(\vec{q}) \dot{q}_j \dot{q}_k - U(\vec{q})$

obecná energie $E = \frac{1}{2} T_{jk}(\vec{q}) \dot{q}_j \dot{q}_k + \hat{U}(\vec{q}) \qquad \mathbb{T} = (T_{kj}(\vec{q}))$ symetrická pozitivně definitní matice

obecná hybnost

$$p_i = \frac{\partial \hat{L}}{\partial \dot{q}_i} = \frac{1}{2} T_{jk} \delta_{ji} \dot{q}_k + \frac{1}{2} T_{jk} \dot{q}_j \delta_{ki} = \frac{1}{2}(T_{ik} \dot{q}_k + T_{ji} \dot{q}_j) = T_{ik} \dot{q}_k \quad \Big/ (\mathbb{T}^{-1})_{ji} \Rightarrow (\mathbb{T}^{-1})_{ji} p_i = (\mathbb{T}^{-1})_{ji} (\mathbb{T})_{ik} \dot{q}_k = \dot{q}_j$$

Hamiltonova funkce

$$H = \frac{1}{2} T_{jk} (\mathbb{T}^{-1})_{ji} (\mathbb{T}^{-1})_{kl} p_i p_l + \hat{U}(\vec{q}) = \frac{1}{2} (\mathbb{T}^{-1})_{ji} \delta_{jl} p_i p_l + U(\vec{q}) = \frac{1}{2} (\mathbb{T}^{-1})_{ij} p_i p_j + \hat{U}(\vec{q})$$