74. ON MULTIPLE PRODUCTION OF PARTICLES DURING COLLISIONS OF FAST PARTICLES

1. General Relations

Collisions of ultra-fast nuclear particles can be accompanied by the appearance of a large number of new particles (many-pronged stars in cosmic radiation). Fermi[1] propounded the ingenious idea of the possibility of applying statistical methods for studying this process. However, the quantitative calculation given by him appears unconvincing to us and incorrect at several points (in particular, in regard to distribution in energy and angle).

Qualitatively the whole process of collision has the following appearance. At the moment of collision there appear a large number of particles† concentrated in a volume whose linear dimensions are determined by the range of the nuclear forces and by the energies of the colliding particles (concerning this, see below); it must be emphasised that we can speak of the number of particles at this moment only in a limited sense, since for a system with such a high density of strongly interacting particles (mesons and nucleons) the concept of the number of particles has in general no precise meaning. The "mean free path" of particles in such a system is clearly very small compared to its dimensions. In the course of time, the system expands, but the aforementioned property of the free path must be valid also for a significant part of the process of expansion. This part of the expansion process must have a hydrodynamic character, since the smallness of the mean free path permits us to consider the motion of the matter in the system in a macroscopic hydrodynamical fashion as the motion of an ideal (non-viscous and non-heat-conducting) liquid. Since the velocities in the system are comparable to the velocity of light, we are dealing, not with ordinary, but rather with relativistic hydrodynamics.

The total "number of particles" in the system is not at all constant during the course of the hydrodynamic stage of the expansion. Therefore, the number of particles in the resulting star is determined, not by the number of particles which appear at the very moment of collision (as Fermi mistakenly assumes) but rather by the number of particles in the system at the moment of transition to the second stage of the expansion—the stage of free separation of the particles. This essential point was first made by I. Ya. Pomeranchuk[2].

Л. Д. Ландау, О множественном образовании частиц при столкновениях быстрых частиц, *Известия Академии Наук СССР, Серия Физическая*, **17**, 51 (1953).

† In fact, the appearance of a large number of particles is the condition for the applicability of the method for treating the problem which is presented below, and of the associated formulas.

The transition from the first stage to the second occurs when the free path of particles in the system becomes equal to its linear dimensions. A very essential point is that at that moment the order of magnitude of the temperature of the system is

$$T_c \sim \mu c^2 \tag{1}$$

(μ is the meson mass; the temperature in always given in energy units), practically independent of the properties of the system, i.e. of the energy of the colliding particles. In fact, for values of the temperature substantically lower than μc^2, the density of the equilibrium number of particles falls exponentially with cooling (as $e^{-\mu c^2/T}$) so that the mean free path rapidly becomes equal to the dimensions of the expanding system, even when the latter are relatively large. Formula (1) for T_c (with the π-meson mass substituted for μ) is also valid when, in addition to mesons, other heavier particles are formed, since in order for the free path of all particles to be small, it is already sufficient that there be a high density of π-mesons in the system.

For the hydrodynamic considerations, it is necessary to have an equation of state for the matter in the system. As equation of state of highly compressed matter for temperatures $T \gg \mu c^2$ we use:

$$p = \frac{\varepsilon}{3}, \tag{2}$$

where p is the pressure and ε is the energy density. Although we have not at present any rigorous proof that this must be the equation of state for arbitrary matter in the ultrarelativistic case, nevertheless in our opinion this assumption is highly plausible.

Since the number of particles in the system is not fixed, but is rather determined from the conditions of statistical equilibrium, its chemical potential (just as for black-body radiation) is

$$\zeta = \varepsilon - T s + p = 0$$

(s is the entropy per unit volume). Then

$$T s = \varepsilon + p = \frac{4\varepsilon}{3}$$

so if we take into account also that for fixed volume (equal to unity) $d\varepsilon = T\,ds$ we find the relations:

$$S \sim \varepsilon^{3/4}; \quad T \sim \varepsilon^{1/4} \tag{3}$$

which, as expected, coincide with the relations for black-body radiation.

The computation of the total number of particles appearing during the break-up is greatly simplified if we consider the motion of the ideal fluid to be adiabatic. The only thing that could destroy the adiabaticity would be shock waves, and it is hard to imagine how they could be formed during the expansion process. Therefore, the entropy of each of the individual regions of the system remains unchanged during the expansion.

Let us break up the system into a set of regions which are macroscopically small, i.e. practically uniform, but which still contain a sufficiently large number of particles; let s_α be the entropies of these regions. Also let n_α be the number of particles in the α-th region which have been produced at the time of the start of its free separation. This time may not be the same for the various regions, since the system as a whole is highly non-uniform. The quantities s_α and n_α individually depend strongly on the temperature (for $T \ll \mu c^2$, they vary as $e^{-\mu c^2/T}$), but the ratio s_α/n_α depends only slightly on temperature, so that, since T_c in turn depends little on the properties of the system, we may consider that

$$n_\alpha = \text{const} \cdot s_\alpha,$$

where the constant ratio is a universal constant [if we measure entropy in dimensionless units, dimensional arguments show that the constant is of order $(\mu c/\hbar)^3$]. Summing this equality over all domains, we find that

$$N = \text{const} \cdot S, \tag{4}$$

where N is the total number of particles in the star, and S is the total entropy of the system. Since the entropy stays constant during the whole course of the hydrodynamic stage of the expansion, we may consider S to be the entropy of the system at the initial time—the time of the collision.† Formula (4) enables us to determine the total number of particles appearing during the collision, without a detailed examination of the motion of the system.

2. Total Number of Particles

Let us first consider "head-on" collisions in which the particles pass each other at distances comparable to the range of interaction, as distinguished from peripheral collisions where the impact parameter is large compared to the range of force.

We start with head-on collisions of two protons, and determine the energy dependence of the total number of particles formed. Let E' be the energy of each of the protons in the centre of mass system (c.m.s). The total entropy of the system, S, is proportional to $\varepsilon^{3/4} V$ where V is the volume over which the energy is distributed. In the c.m.s the matter is at rest at the moment immediately following the collision. Therefore, $\varepsilon = E'/V$, and so the entropy, and consequently the number of particles, is proportional to $E'^{3/4} V^{1/4}$.

The transverse dimension of the system, a, is of order of magnitude of the range of nuclear force, i.e. $a \sim \hbar/\mu c$. The longitudinal dimension (in the c.m.s.) is shortened by the Lorentz contraction in the ratio $\sim Mc^2/E'$ (M is the

† More precisely, after the passage of shock waves, which can arise at the moment of collision; the passage of a shock wave is accompanied by a compression of the matter, after which the expansion stage begins and proceeds adiabatically from then on.

proton mass). Thus the system is in the form of a highly flattened disk, and its volume is

$$V \sim a^3 M c^2/E'.$$

So the number of particles is

$$N \sim E'^{3/4} V^{1/4} \sim \sqrt{E'}$$

or, going over to the energy E in the laboratory system in which one of the protons is at rest, using the formula $E M c^2 = 2E'^2$, we finally get:

$$N \sim E^{1/4}$$

This formula coincides with the one obtained by Fermi, but his reasoning appears to us to be completely unconvincing. From dimensional arguments (and taking account of the fact that the ratio of masses of proton and π-meson is fairly close to the unity) we may write:

$$N = K\left(\frac{E}{2 M c^2}\right)^{1/4} \tag{5}$$

where K is a constant of the order of unity.

Now let us consider the collision of two identical nuclei of atomic weight A. It would be completely erroneous to treat such a collision as a series of collisions of nuclear protons and neutrons. In fact, since the distance between nuleons in the nuclei is precisely of the order of their range of interaction, we must look upon the result of the collision as a process of meson formation involving as a unit the whole space occupied by the nuclei.

Suppose that the speed of the incident nucleus is equal to that of the proton in the preceding problem. Then its energy will be A times as large. Since the mass density in a nucleus is approximately the same as that of the proton, referred to its sphere of interaction, the energy density immediately after collision is the same as in the previous case. Since the Lorentz contraction is unchanged, the number of particles formed is simply proportional to the volume of the nucleus, i.e. to A. Thus we finally obtain:

$$N = K A\left(\frac{E}{2 A M c^2}\right)^{1/4} = K A^{3/4}\left(\frac{E}{2 M c^2}\right)^{1/4} \tag{6}$$

For a given energy, the number of particles is proportional to $A^{3/4}$. We note that according to this formula, heavy nuclei are much more effective in particle formation than protons: two nuclei with energy E give as many particles as two protons with energy $E A^3$.

When the two nuclei have different weights the problem becomes more complicated, but elementary considerations related to the fact that in collision the lighter nucleus pulls out only a part of the heavier one, show that the number of particles is determined essentially by the mass of the lighter nucleus, and depends only slightly on the mass of the heavier one.

If we are dealing with collisions of a meson with a nucleon or nucleus, it follows that we should expect relatively little difference from the case of a nucleon.

Determination of the constant K from existing experimental data gives the value

$$K \sim 2.$$

As for peripheral collisions of the nucleons, at first glance one might conclude that the average number of particles produced should decrease rapidly with increasing impact parameter. A basis for this conclusion might be the fact that the rest energy of the matter concentrated in each individual region of the meson field of the colliding nucleons decreases rapidly (exponentially) with increasing distance from their "centre". However, the incorrectness of this derivation is clear from the fact that it leads to a contradiction with the quantum uncertainty relations; the rest energy of a portion of the system would turn out to small compared to the uncertainty

$$\Delta E \sim \hbar c / \Delta,$$

where Δ is the thickness of the region, compressed by the Lorentz contraction just as for central collisions. In fact, this relation means only that the quantity which is small is not the actual energy of the system (in those cases where such a system occurs at all) but rather its mathematical expectation. In other words, it is not the number of particles appearing that decreases, but only the probability that such a collision shall occur.

Thus for collisions of two nucleons it is in general meaningless to distinguish between central and peripheral collisions; the effective cross-section for collision with production of a many-pronged star is determined by the "radius" of the nucleon, $\hbar/\mu c$†. The picture is somewhat changed in the case of a collision of two nuclei. It is clear that as we vary the impact parameter from zero to the sum of the radii of the nuclei, the number of particles formed must decrease from the maximum value given by formula (6) to the value given by (5) and corresponding to the collision of two nucleons.

3. Distribution of Particles Produced in Energy and Direction

A study of the angular distribution of the particles formed, and their distribution in energy, requires a detailed consideration of the hydrodynamical motion of the matter in the system.

The relativistic hydrodynamic equations are contained in the relations

$$\frac{\partial T^{ik}}{\partial x^k} = 0, \qquad (7)$$

† This result was clarified in discussions with E. L. Feinberg.

where T^{ik} is the energy-momentum tensor of the matter:

$$T^{ik} = p\, g^{ik} + (\varepsilon + p)\, u^i u^k \tag{8}$$

(u^i is the four-velocity; $g^{11} = g^{22} = g^{33} = 1$, $g^{00} = -1$; from now on we set $c = 1$).

As we have already indicated, at the moment of collision the system has the form of a highly flattened disk. This shape is maintained throughout a significant part of the hydrodynamical stage of the expansion. During this stage, the motion of the matter can be considered to be one-dimensional, along the short axis of the disk (x-axis). Then the equations of motion are:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{01}}{\partial x} = 0, \quad \frac{\partial T^{01}}{\partial t} + \frac{\partial T^{11}}{\partial x} = 0, \tag{9}$$

where

$$T^{00} = \varepsilon (u^0)^2 + p(u^1)^2, \quad T^{01} = (\varepsilon + p)\, u^0 u^1, \quad T^{11} = \varepsilon (u^1)^2 + p(u^0)^2, \tag{10}$$

and u^0 and u^1 are related by the equation:

$$(u^0)^2 - (u^1)^2 = 1. \tag{11}$$

In the c.m.s. the "disk" expands symmetrically to both sides. We choose our co-ordinate origin in the median plane and shall consider the motion in the half-space expanding along the positive x-axis (so that $\dot{x} > 0$, $u^1 > 0$).

Let us call the initial thickness of the "disk" Δ. We consider some instant of time $t \gg \Delta$, when the expansion has already progressed significantly. Neglecting the initial thickness of the disk we can assert that all the matter will be in the region $0 < x < t$, since the velocity cannot exceed that of light. Most of this space will contain matter which, though moving with a speed comparable to the light velocity, is not ultra-relativistic; only in a thin layer $t - x \ll t$ will there be matter moving with a velocity close to that of light. As we shall see later, in this last region there is concentrated only a *small* part of the entropy, but a *large* part of the energy of the system. Therefore, the examination of this small-sized ultra-relativistic region is very essential. To do this we replace the variable x by $\xi = t - x$. Then the first of equations (9) takes the form:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial (T^{00} - T^{11})}{\partial \xi} = 0, \tag{12}$$

and, subtracting (12) from the second equation of (9) we find:

$$\frac{\partial}{\partial t}(T^{00} - T^{11}) + \frac{\partial}{\partial \xi}(T^{00} - 2T^{01} + T^{11}) = 0. \tag{13}$$

In the ultra-relativistic case both components u^0, u^1 of the four-velocity are large compared to unity and almost equal (we recall that $u^0 = 1/\sqrt{1 - v^2}$ $u^1 = v/\sqrt{1 - v^2}$, where v is the ordinary velocity (in units $c = 1$). Later we shall denote by u (in first approximation) either of the quantiies u^0 and u^1.

According to (10):

$$u^0 \approx u^1 = u \gg 1, \quad u^0 - u^1 \approx \frac{1}{2u}.$$

Using these equalities and the equation of state (2), we obtain from (10):

$$\left.\begin{aligned} T^{00} &\approx (\varepsilon + p)\, u^2 = \frac{4}{3}\varepsilon u^2, \\ T^{00} - T^{01} &= (\varepsilon u^0 - p u^1)(u^0 - u^1) \approx \frac{\varepsilon}{3}, \\ T^{00} - 2T^{01} + T^{11} &= (\varepsilon + p)(u^0 - u^1)^2 \approx \frac{\varepsilon}{3u^2}, \end{aligned}\right\} \tag{14}$$

after which equations (12–13) take the form:

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\varepsilon u^2) &= -\frac{1}{4}\frac{\partial \varepsilon}{\partial \xi}, \\ \frac{\partial \varepsilon}{\partial t} &= -\frac{\partial}{\partial \xi}\left(\frac{\varepsilon}{u^2}\right). \end{aligned}\right| \tag{15}$$

We shall look for solutions of these equations in the domain of values $t \gg \xi \gg \Delta$.

A solution satisfying all the necessary requirements can be obtained as follows. Let us make the assumption, which we later show to be valid, that the function $u(\xi, t)$ is such that

$$u^2 = f\frac{t}{\xi}, \tag{16}$$

where f is a slowly (logarithmically) varying function of ξ and t. Neglecting the derivatives of f, we then obtain from (15)

$$f\frac{\partial}{\partial t}(\varepsilon t) = -\frac{\xi}{4}\frac{\partial \varepsilon}{\partial \xi}, \quad f t\frac{\partial \varepsilon}{\partial t} = -\frac{\partial}{\partial \xi}(\varepsilon \xi).$$

Next we introduce the new variables

$$\tau = \ln\frac{t}{\Delta}, \quad \eta = \ln\frac{\xi}{\Delta}, \tag{17}$$

and in place of ε, a new unknown function φ according to the relation

$$\varepsilon = e^{\varphi}. \tag{18}$$

From the two equations thus obtained,

$$f\left(1 + \frac{\partial \varphi}{\partial \tau}\right) = -\frac{1}{4}\cdot\frac{\partial \varphi}{\partial \eta} \quad \text{and} \quad f\frac{\partial \varphi}{\partial \tau} = -\left(1 + \frac{\partial \varphi}{\partial \eta}\right), \tag{19}$$

we eliminate f and get:

$$1+\frac{\partial\varphi}{\partial\tau}+\frac{\partial\varphi}{\partial\eta}+\frac{3}{4}\cdot\frac{\partial\varphi}{\partial t}\cdot\frac{\partial\varphi}{\partial\eta}=0. \tag{20}$$

Following the general procedure for obtaining the general integral of a partial differential equation of first order, we first form the complete integral:

$$\varphi=A\,\eta-\frac{4(1+A)}{4+3A}\tau+B, \tag{21}$$

containing two constants A and B. The general integral (containing one arbitrary function) is obtained from the complete integral if we consider B to be a function of A, determined by equation (21) and the equation

$$\eta-\frac{4}{(4+3A)^2}\tau+\frac{\mathrm{d}B}{\mathrm{d}A}=0 \tag{22}$$

obtained by setting the derivative $\partial\varphi/\partial A$ equal to zero.

Since we are looking for a solution of the equation of motion in the region of values $t \gg \Delta$, $\xi \gg \Delta$, the "initial moment" of the motion corresponds to values $\tau\sim 1$, $\eta\sim 1$. At this "moment", the system in the domain under consideration can be regarded as still uniform, so that the function φ is practically constant and equal to some value φ_0 (the logarithm of the initial energy density ε_0). Thus the initial condition for our problem, to within the logarithmic accuracy we are using, is:

$$\varphi-\varphi_0\sim 1 \quad \text{for} \quad \eta\sim 1, \quad \tau\sim 1. \tag{23}$$

A solution satisfying this condition is obtained from (21–22), if the arbitrary function $B(A)$ is chosen so that

$$B-\varphi_0\sim 1, \quad B'\sim 1.$$

Then we can omit B' altogether in (22), (since $\eta \gg 1$, $\tau \gg 1$), and can set $B\approx\varphi_0$ in (21). We then have from (22)

$$\frac{4+3A}{2}=\sqrt{\frac{\tau}{\eta}}$$

(we choose the positive root, since in the other case the function f in (16) would turn out to be negative, which is clearly impossible), after which (21) gives:

$$\left.\begin{aligned}\varphi&=\varphi_0-\frac{4}{3}\left(\eta+\tau-\sqrt{\tau\eta}\right),\\ \varepsilon&=\varepsilon_0\exp\left[-\frac{4}{3}\left(\eta+\tau-\sqrt{\tau\eta}\right)\right].\end{aligned}\right\} \tag{24}$$

When ξ becomes of the order of t, formula (16) as expected gives $u \sim 1$. From formula (24) it follows that in this region ($\eta \approx t$):

$$\varepsilon = \varepsilon_0 \, e^{-4\tau/3} = \varepsilon_0 \left(\frac{\Delta}{t}\right)^{4/3}$$

Even though the domain $\xi \sim t$ is outside the region of ultra-relativistic motion, this result should be correct as to order of magnitude.

The function f is found from φ using either of the equations (19):

$$f = \frac{1}{2}\sqrt{\frac{\tau}{\eta}} \, .$$

In accordance with our assumptions, it is a slowly varying function of t and ξ, of order unity.

Using the formulas we have obtained, let us see how the energy and entropy are distributed throughout the thickness of the "disk". The energy density is given by the component $T^{00} \sim \varepsilon u^2$ of the energy–momentum tensor (we recall that for each element of the matter ε is the energy density in the "proper" frame of reference, in which that element is at rest). So for the energy dE located in a slab of thickness $d\xi$ we have:

$$dE \sim \varepsilon a^2 u^2 \, d\xi = \varepsilon a^2 u^2 \xi \, d\eta ,$$

where a is the radius of the disk. Setting $u^2 \sim t/\xi$, in accord with (16) and using equation (24) we obtain:

$$dE \sim \exp\left[-\frac{1}{3}\left(\sqrt{\tau} - 2\sqrt{\eta}\right)^2\right] d\eta . \qquad (25)$$

From this it is clear that the energy distribution has a maximum at $\eta = \tau/4$; this means that the energy is concentrated mainly in the region

$$\xi \sim \sqrt[4]{t \Delta^3} \, .$$

For $t \gg \Delta$ we get $\xi \ll t$, so that this region is at the limit of applicability of the one-dimensional solution we are considering.

The entropy density is given by the fourth component s^0 of the four-vector of entropy current density $s^i = s u^i$. Since $s \sim \varepsilon^{3/4}$ (according to (4)), $s^0 \sim u \varepsilon^{3/4}$, and we find for the entropy associated with a slab of thickness $d\xi$:

$$dS \sim s u a^2 \, d\xi \sim a^2 s u \xi \, d\eta ,$$

or, using formula (24):

$$dS \sim \exp\left[-\frac{1}{2}\left(\sqrt{\tau} - \sqrt{\eta}\right)^2\right] d\eta . \qquad (26)$$

This distribution has a maximum for $\eta = t$; i.e. the entropy unlike the energy is concentrated mainly in the region $\xi \sim t$.

The solution of the equation of motion which we have obtained is applicable so long as the angle of flight θ (the angle which the trajectory of a given element of the matter makes with the x-axis) is sufficiently small. This is necessary in order that the distance $t\,\theta$, which the element travels during the time t in the transverse direction, be small compared to the transverse dimensions of the system, a:

$$t\theta \ll a. \tag{27}$$

To evaluate the small angle θ, we use the transverse components of equation (7), which we have as yet not considered. Thus we get:

$$\frac{\partial T^{02}}{\partial t} \sim \frac{\partial T^{22}}{\partial y},$$

or, to order of magnitude,

$$\frac{T^{02}}{t} \sim \frac{T^{22}}{a};$$

so that substituting $T^{02} \sim \varepsilon\, u^2\, \theta$ and $T^{22} \sim \varepsilon$ (the transverse component of the four-velocity is $u^y \sim u\,\theta$), we get:

$$u^2\,\theta \sim \frac{t}{a}.$$

Finally, noting that $u^2 \sim t/\xi$, we find:

$$\theta \sim \frac{\xi}{a}. \tag{28}$$

Combining this formula (27) we see that the condition for applicability of the one-dimensional solution is:

$$t\,\xi \ll a^2. \tag{29}$$

We note that the limiting time for the one-dimensional solution is the greater, the smaller the value of ξ. For the central region, $\xi \sim t$, and the limiting time is $t \sim a$.

Starting at the moment

$$t_1 = \frac{a^2}{\xi}, \tag{30}$$

a significant sideways motion appears in the hydrodynamical motion; we shall call the resulting motion of the matter conical hydrodynamic flight. As we shall see later, in this stage of the motion the velocity approaches that of light so quickly, that for each element of matter the quantity ξ remains practically constant in time. In addition, one can show that all derivatives of hydrodynamic quantities, both with respect to the direction of ξ as well as with respect to the transverse direction, can be neglected in the equations. Thus, in particular, it follows that, because of the smallness of the sidewise forces, the direction of motion will remain unchanged, i.e. the flight will proceed radially (conically).

Furthermore, in view of the smallness of the forces during conical flight, the energy flux traveling within any cone, $\theta = \text{const}$, must remain constant; the same is true for the entropy flow. The cross sectional area of such a cone is proportional to t^2, so the conditions of constancy of flow of energy and entropy are:

$$\varepsilon\, u^2\, t^2 = \text{const}, \quad s\, u\, t^2 \sim \varepsilon^{3/4}\, u\, t^2 = \text{const}. \tag{31}$$

From these two relations we find:

$$u \sim t, \quad \varepsilon \sim \frac{1}{t^4}, \tag{32}$$

which give the law of variation with time of u and ε during conical flight. From (32) we see that in this case the velocity actually approaches that of light faster than during the preceding stage. The change in the co-ordinate ξ of the moving element of matter is given by the formula:

$$\frac{d\xi}{dt} = 1 - v_x = (u^0 - u^1)\sqrt{1 - v^2} \approx \frac{1}{2u^2} \sim \frac{1}{t^2},$$

from which it is clear that during this stage of the flight, the quantity approaches a constant value more rapidly.

For $t \sim t_1$, the solution (32) must agree to order of magnitude with the one-dimensional solution considered earlier. For the "joining" of the two solutions, it is convenient to introduce the symbols λ and L, according to the equations:

$$\frac{\xi}{a} = e^{-\lambda}, \quad \frac{\Delta}{a} = e^{-L}. \tag{33}$$

Then

$$\eta = \ln\frac{\xi}{\Delta} = L - \lambda, \tag{34}$$

while the value of the variable τ corresponding to the moment t_1 is

$$\tau_1 = \ln\frac{t_1}{\Delta} = \ln\frac{a^2}{\xi\Delta} = L + \lambda. \tag{35}$$

Substituting this value in (26), we find that the entropy distribution is given by

$$dS \sim e^{\sqrt{L^2 - \lambda^2}}\, d\lambda.$$

Since each element of the matter now moves with $\xi = \text{const}$ while its entropy, by virtue of the adiabaticity of the motion, remains constant, the same formula gives the entropy at the moment of break-up of the matter into individual freely moving particles. The number of particles produced will be distributed according to this same law:

$$dN = C e^{\sqrt{L^2 - \lambda^2}}\, d\lambda, \tag{36}$$

where C is a normalising factor. The angle of flight

$$\theta \sim \frac{\xi}{a} = e^{-\lambda} \tag{37}$$

remains constant along with ξ for each element of the matter, and consequently for each particle. Consequently, formulae (36) and (37) determine in parametric form (parameter λ) the angular distribution of the produced particles (in the c.m.s.). The constant L which appears in the formula is related simply to the energy of the colliding particles. In fact, the ratio Δ/a is the Lorentz contraction of the system and is equal, in the notation of section 2 to

$$\frac{MAc^2}{E'} = \sqrt{\frac{2MAc^2}{E}}$$

(where MA is the mass of the particles). Therefore

$$L = \frac{1}{2}\ln\frac{E}{2MAc^2}. \tag{38}$$

The distribution (36) shows that, although the angle of departure in the c.m.s. is of the order of unity for most of the particles, there also occur much smaller angles. It is easy to see that the angular distribution does not at all show spherical symmetry, as Fermi assumed, but that $\mathrm{d}N/\mathrm{d}o$, referred to unit solid angle, increases rapidly with decreasing θ.

Formula (36) is easily written in explicit form. In order to take into account angles of the order of unity, we define λ as

$$\lambda = -\ln\tan\frac{\theta}{2}. \tag{39}$$

With this definition, the smallest value $\lambda = 0$, corresponds to the largest possible value $\theta = \pi/2$. Formula (36) then becomes

$$\mathrm{d}N \sim \exp\left[\sqrt{L^2 - \ln^2\tan\frac{\theta}{2}}\right]\frac{\mathrm{d}o}{\sin^2\theta}. \tag{40}$$

This formula agrees well with the experimental data[3]. For practical purposes, formula (36) can be written to sufficient accuracy in the form:

$$\mathrm{d}N \sim \mathrm{e}^{-\frac{\lambda^2}{2L}}\mathrm{d}\lambda. \tag{41}$$

Thus the angular distribution can be written as a Gaussian distribution, if we choose as variable the quantity $\lambda = -\ln\tan\theta/2$. In view of the logarithmic dependence of λ on θ, the actual distribution curve of the particles with respect to the angle θ itself must have relatively very straight tails on both sides of the maximum.

We note that the largest value of λ which it is still meaningful to consider must correspond to the condition:

$$\int_{\lambda=\lambda_{max}}^{\lambda=L} dN \sim 1,$$

or, substituting (36), to logarithmic accuracy,

$$C e^{\sqrt{L^2-\lambda_{max}^2}} \sim 1.$$

According to formulae (5) and (36) the total number of particles is $N \sim e^{L/2}$; therefore

$$\int_{\lambda=0}^{\lambda=L} dN \sim c\, e^{L} \sim e^{L/2}.$$

Thus $C \sim e^{L/2}$, and we obtain for λ_{max},

$$\lambda_{max} = \frac{\sqrt{3}}{2} L. \tag{42}$$

For determining the energy distribution of the particles, we consider the quantity u, which is proportional to the energy of the particles (the energy of a particle is the time component of the four-vector μu^i: $\mu u^0 \approx \mu u$). During the stage of one-dimensional motion $u \sim \sqrt{t/\xi}$, and at the moment $t = t_1$ it reaches the value $u \sim \sqrt{t_1/\xi}$. Therefore, "tacking on" the one-dimensional motion to the solution (32), we find that during the stage of conical motion:

$$u \sim \sqrt{\frac{t_1}{\xi}} \cdot \frac{t}{t_1} \sim \frac{t}{a}. \tag{43}$$

In similar fashion we match the laws (24) and (32) of variation of the "proper" energy density ε. For $t \sim t_1$ the quantity ε reaches the value:

$$\varepsilon = \varepsilon_0 \exp\left[-\frac{4}{3}\left(2L - \sqrt{L^2-\lambda^2}\right)\right].$$

Determining from this the coefficient of proportionality in (32), we find:

$$\varepsilon = \varepsilon_0 \left(\frac{t_1}{t}\right)^4 \exp\left[-\frac{4}{3}\left(2L - \sqrt{L^2-\lambda^2}\right)\right]. \tag{44}$$

The start of the free separation of the particles corresponds to the moment t_0 when ε, decreasing, reaches the value ε_c corresponding to the criterion (1). From (44) we find:

$$t_c \sim t_1 \left(\frac{\varepsilon_0}{\varepsilon_c}\right)^{1/4} \exp\left[-\frac{1}{3}\left(2L - \sqrt{L^2-\lambda^2}\right)\right].$$

Setting $t \sim t_c$ in (43) and substituting for t_1 from (35), we find the following expression for the energy $\mu u'$ of the particles at the moment of their free separation:

$$\mu u' \sim \mu \frac{t_c}{a} = \text{const}\, \exp\left[\lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right].$$

We note that the energy of the outgoing particles is measured by the ratio of the time (or the distance from the origin) at the moment of decay into particles to the characteristic time a/c of the system. The constant coefficient in the expression for $\mu u'$ is determined from the obvious relation:

$$\int \mu u' \,\mathrm{d}N = E' \sim \sqrt{EMA} \sim MA\,\mathrm{e}^L,$$

and we get finally:

$$\mu u' \sim M \exp\left\{-\frac{L}{6} + \lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\}. \tag{45}$$

Formulae (36) and (45) give in parametric form the energy distribution (in the c.m.s.) of the particles produced. From (45) we see that most of the particles ($\lambda \sim 0$) have energies $\mu u' \sim M\,\mathrm{e}^{L/6} \sim M(E'/AM^{1/6})$ only slightly exceeding M.

We must still go over from the c.m.s. to the original laboratory frame of reference in which one of the nucleons was at rest before the collision. The angle χ of the outgoing particle in the laboratory system is related to the angle θ in the c.m.s. by the transformation formula:

$$\tan\chi \approx \chi = \frac{v'\sqrt{1 - V^2}\sin\theta}{v'\cos\theta + V},$$

where v' is the velocity of the particle in the c.m.s., and V is the velocity of the c.m. relative to the laboratory system. We may immediately write $v' = 1$ in the numerator, and in the denominator, write:

$$v'\cos\theta + V \approx v'(1 + \cos\theta) + \frac{1}{2}(V^2 - v'^2),$$

or, since V is closer to unity than v':

$$v'\cos\theta + V \approx 1 + \cos\theta + \frac{1}{2}(1 - v'^2) = 1 + \cos\theta + \frac{1}{2u'^2}.$$

The last term on the right can be neglected for all cases except when θ is too close to π. However, it is easy to see that the angles we have found satisfy the inequality $\theta,\ \pi - \theta \gg 1/u'$; this is equivalent to the inequality:

$$\exp\left\{\frac{L}{6} - \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\} \ll 1$$

(according to (39) and (45), which is actually satisfied for all $\lambda < \lambda_{\max}$. Thus we can set $v' \cos\theta + V \approx 1 + \cos\theta$, and the formula for transforming angles to the laboratory system takes the form:

$$\chi = \sqrt{1 - V^2} \tan\frac{\theta}{2}. \tag{46}$$

In this connection we note the following curious fact. Independently of any detailed computations, the distribution of outgoing particles, for a collision of two identical particles, is symmetric in the c.m.s., i.e. angles θ occur just as often as $\pi - \theta$. Since $\tan(\pi - \theta)/2 = 1/\tan(\theta/2)$, it follows automatically that, upon averaging over all particles,

$$\overline{\ln\chi} = \ln\sqrt{1 - V^2} = -L. \tag{47}$$

In other words, the geometrical mean of all the angles of separation gives just the value of the velocity of the c.m. and, consequently, the velocities of the incident particles (for a collision of two identical particles).

Substituting the value $\tan(\theta/2) = e^{-\lambda}$ and $\tan(\pi - \theta)/2 = e^{-\lambda}$ in (46) for particles moving in opposite directions in the c.m.s., we obtain:

$$\chi = e^{-L \mp \lambda}.$$

This formula has the special property that when we change from particles going to the right in the c.m.s. to particles travelling to the left, there is merely a change in sign of the quantity λ. We can therefore write

$$\chi = e^{-L-\lambda}, \tag{48}$$

and consider formulae (36) and (48) as giving the angular distribution of all particles in the laboratory system, where λ can take both positive and negative values.

For the transformation of the energy of particles moving to the right, we have

$$u \sim \frac{u'}{\sqrt{1 - V^2}} = e^L u',$$

and for particles moving to the left we get (noting that $\theta \gg 1/u$):

$$u \sim \frac{\theta^2 u'}{\sqrt{1 - V^2}} = u' e^{L - 2\lambda}.$$

Substituting (45), this gives:

$$\mu u \sim M \exp\left\{\frac{5L}{6} \pm \lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\}$$

This formula too has the property that it describes particles moving to both the right and the left in the c.m.s., if we write

$$\mu u \sim M \exp\left\{\frac{5L}{6} + \lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\}, \tag{49}$$

and give λ both signs.

Formulae (36) and (49) give the energy distribution of the particles in the laboratory system. The coefficients in these formulas can be made more precise if we use the obvious relations:

$$\int \mathrm{d}N = N, \quad \int \mu u \,\mathrm{d}N = E.$$

In the integrations we can, to the accuracy we are considering, expand the exponent in a series in the neighbourhood of the maximum. We then get

$$\mathrm{d}N = \frac{N}{\sqrt{2\pi L}} e^{\sqrt{L^2-\lambda^2}} \mathrm{d}\lambda,$$

or, taking account of (6) and (38),

$$\mathrm{d}N = \frac{KA}{\sqrt{2\pi L}} e^{-\frac{L}{2} + \sqrt{L^2-\lambda^2}} \mathrm{d}\lambda. \tag{50}$$

It is understood that the coefficient in this formula is actually a slowly varying (non-exponential) function of the ratio λ/L. For the energy we get:

$$\mu u = \frac{5\sqrt{5}}{2\sqrt{3}} \cdot \frac{M}{K} \exp\left\{\frac{5L}{6} + \lambda + \frac{1}{3}\sqrt{L^2-\lambda^2}\right\} \tag{51}$$

Here the same remarks apply to the coefficient as were made in the last case. From formula (51) it is clear that most of the particles have an energy of order

$$Mc^2\left(\frac{E}{2MAc^2}\right)^{7/12}$$

in the laboratory system.

We note that both the angular and energy distributions of the particles are close to Gaussian if we use the logarithm of these quantities as variables; consequently, they have quite straight tails on both sides of the maximum. The results of a computation based on (51) are shown in Fig. 1.

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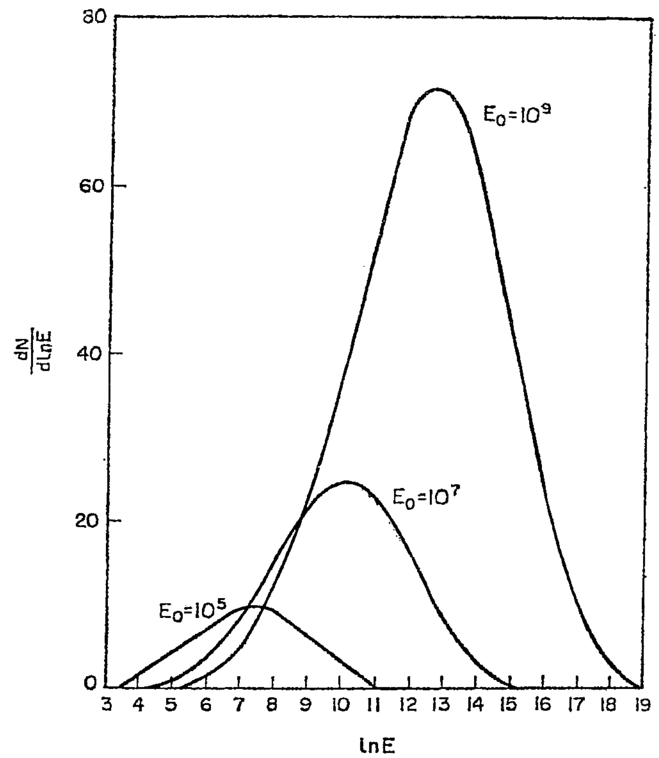

FIG. 1.

Differential energy spectra of secondary particles during nuclear interactions at high energy (for varying energies E_0 of the initial particles). The areas under the curves are proportional to the total number of secondary particles (mesons and nucleons).

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